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Volume 4, Number 3, February 2018

Group 5: mathematics

Understanding differentiation

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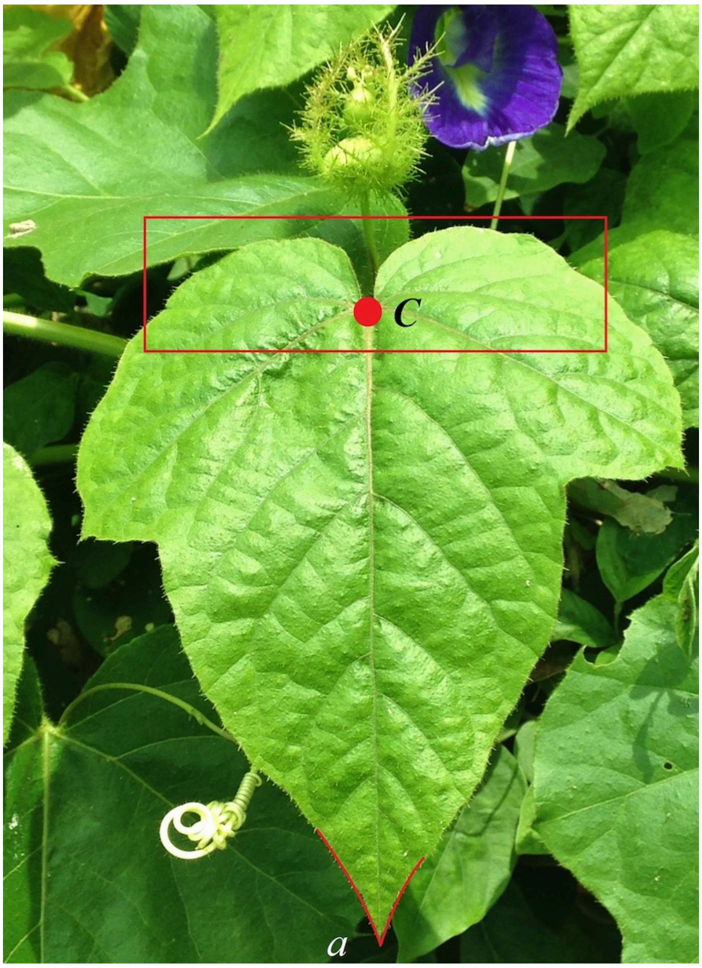
Answers to the questions on p. 30 of the magazine.

1 **a** Since $P = 200 + 50t + t^2$, where t is measured in minutes then $\frac{dP}{dt} = 50 + 2t$. The value of $\frac{dP}{dt}$ evaluated at $t = 10$ is 70. That is the growth rate at 10 minutes is an increase of 70 bacteria per minute.

b Since $Q = 980 - 150P$, where Q is the quantity demanded in thousands in a week and P is the price in USD then $\frac{dQ}{dP} = -150$. Before we can calculate the price elasticity of demand using $\frac{dQ}{dP} \times \frac{P_0}{Q_0}$, we need to find out the value of Q_0. $Q_0 = 980 - 150(4) = 380$. The price elasticity of demand at USD 4 is $(-150) \times \frac{4}{380} = -\frac{30}{19}$

2 If we limit the domain of the function (as represented by the edge of the leaf) to the width of the red rectangular box in the photo then the point C is a cusp:



3 We will assume that $(x_0, y(x_0))$ is a local minimum and that the derivative exists at the local maximum. Our job is to show that this derivative is zero to complete the proof.
Since $(x_0, y(x_0))$ is a local minimum then $y(x_0) \leq y(x)$ for all *x* in [*a*,*b*].
Thus, $y(x) - y\ (x_0) \geq 0$, for all *x* in [*a*,*b*]. (1)

Since the derivative exists then

$\lim\limits_{h \to 0^-} \frac{y(x_0+h)-y(x_0)}{h} = \lim\limits_{h \to 0} \frac{y(x_0+h)-y(x_0)}{h} = \lim\limits_{h \to 0^+} \frac{y(x_0+h)-y(x_0)}{h}$.
Let us first consider

$\lim\limits_{h \to 0^-} \frac{y(x_0+h)-y(x_0)}{h}$

Since *h* tends to zero from the left then *h* < 0. The numerator is non-negative from (1). These two implications means that

$\lim\limits_{h \to 0^-} \frac{y(x_0+h)-y(x_0)}{h} \leq 0.$
Next, we consider

$\lim\limits_{h \to 0^+} \frac{y(x_0+h)-y(x_0)}{h}$.

Since *h* tends to zero from the right then *h* > 0. The numerator is non-negative. Thus, we have

$\lim\limits_{h \to 0^-} \frac{y(x_0+h)-y(x_0)}{h} \geq 0.$
Taking these two results together, we can conclude that $\lim\limits_{h \to 0} \frac{y(x_0+h)-y(x_0)}{h} = 0$
Thus, if $(x_0, y(x_0))$ is a local minimum then $y'(x_0) = 0$

4 We can assume the function *y* is continuous on the closed interval [*a*,*b*], $y'(x_0 - \delta) < 0$ and $y'(x_0 + \delta) > 0$ and our job is to show that $y(x_0) \leq y(x)$ for all *x* in the neighborhood of x_0.
Since $y'(x_0 - \delta) < 0$ then the function *y* is a decreasing function such that $y(x_0 - \delta) \leq y(x_0 - 2\delta)$.
Since $\delta > 0$ and *f* is continuous on the closed interval [*a*,*b*] then $y(x_0 - 0) \leq y(x_0 - \delta)$.
Thus, $y(x_0) \leq y(x_0 - \delta)$ for $\delta > 0$.
We now consider $y'(x_0 + \delta) > 0$.
Since $y'(x_0 + \delta) > 0$ then using the similar argument as above *y* is an increasing function such that $y(x_0 + \delta) \leq y(x_0 + 2\delta)$.
Since $\delta > 0$ and *f* is continuous on the closed interval [*a*,*b*] then $y(x_0 + 0) \leq y(x_0 + \delta)$.
Thus, $y(x_0) \leq y(x_0 + \delta)$ for $\delta > 0$.
Taking these two results together we have the desired $y(x_0) \leq y(x)$ for all *x* in $[x_0 - \delta,\ x_0 + \delta]$.
Thus, $(x_0, y(x_0))$ is a local minimum.

5 **a** The procedure for optimization is as follow:
Identify the objective function, that is the function that needs to be maximized or minimized. In this case, the objective function is explicitly given as *f* and our job is easy.
Take the derivative of *f* with respect to the desired variable *x* to find the *x*-coordinates of stationary points.

$f'(x) = x^2 + 3x - 10$

$f'(x) = (x-2)(x+5)$

Set $f'(x) = 0$ and solve for *x*.

$(x-2)(x+5) = 0$

$x = 2$ or $x = -5$.
Substitute these x values back to the original function f to find the corresponding y-coordinates.

$f(2) = \frac{-4}{3}$ and $f(-5) = 55\frac{5}{3}$

The stationary points are thus $(2, \frac{-4}{3})$ and $(-5, 55\frac{5}{3})$.

b This response is essentially the final part of the optimisation procedure that verifies the nature of the stationary points.

From the stationary points above it will seem that $(2, \frac{-4}{3})$ is the local minimum and $(-5, 55\frac{5}{3})$ is the local maximum. A graphic display calculator (GDC) can be used to verify the nature of these stationary points.

Having said so, the question asks us to use an appropriate test. Since the power rule can be used to take the derivative of the quadratic function in $f'(x)$ with ease then we can apply the second derivative test:

$f''(x) = 2x + 3$

Since $f''(2) = 7$ and it is positive then by the second derivative test $(2, \frac{-4}{3})$ is a local minimum point.

Since $f''(-5) = -7$ and it is negative then by the second derivative test $(-5, 55\frac{5}{3})$ is a local maximum point.

Instead of the second derivative test, a first derivative test can also be used.

Since $f'(1.9) = (1.9 - 2)(1.9 + 5) < 0$ and $f'(2.1) = (2.1 - 2)(1.9 + 5) > 0$ then the sign changes from negative to positive in the neighbourhood of $(2, \frac{-4}{3})$. By the first derivative test, $(2, \frac{-4}{3})$ is a local minimum point.

Since $f'(-5.1) = (-5.1 - 2)(-5.1 + 5) > 0$ and $f'(-4.9) = (-4.9 - 2)(-4.9 + 5) < 0$ and then the sign changes from positive to negative in the neighbourhood of $(-5, 55\frac{5}{3})$. By the first derivative test, $(-5, 55\frac{5}{3})$ is a local maximum point.