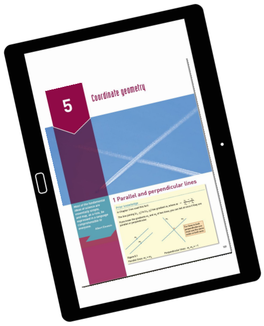

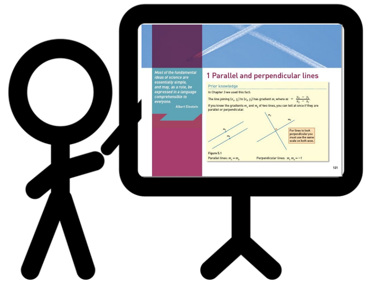

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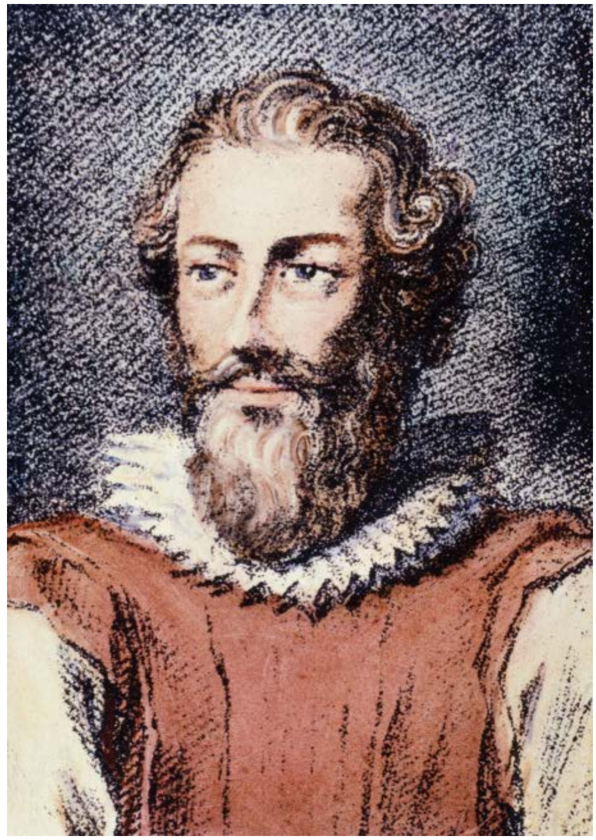
1 Algebraic manipulation



One really can't argue with a mathematical theorem.

Stephen Hawking (1942–2018)

Early mathematics focused principally on arithmetic and geometry. However, in the sixteenth century a French mathematician, François Viète, started work on 'new algebra'. He was a lawyer by trade and served as a privy councillor to both Henry III and Henry IV of France. His innovative use of letters and parameters in equations was an important step towards modern algebra.

Algebra is the foundation of much of what is referred to as 'Pure mathematics'.

Discussion points

- ➜ What is a fraction in arithmetic?
- ➜ What about in algebra?

Simplifying algebraic fractions

Fractions in algebra obey the same rules as fractions in arithmetic. These cover two pairs of operations: × and ÷, and + and −.

Discussion points

- ➜ When can you cancel fractions in arithmetic?
- ➜ What about in algebra?
- ➜ What is a factor in arithmetic?
- ➜ What about in algebra?

Example 1.1

Simplify the following.

(i) $\frac{18}{24}$ (ii) $\frac{2x+2}{3x+3}$

(iii) $\frac{a^2-a-6}{a^2-8a+15}$

Prior knowledge

This chapter is about basic algebra. Although you will have met the work at GCSE you really do need to be fluent and accurate in it.
This chapter is a good opportunity to practise and polish your skills.

Solution

(i) $\frac{18}{24} = \frac{{}^1\cancel{6}\times 3}{{}_1\cancel{6}\times 4} = \frac{3}{4}$

(ii) $\frac{2x+2}{3x+3} = \frac{2\cancel{(x+1)}}{3\cancel{(x+1)}} = \frac{2}{3}$

Remember that in algebra you must factorise before cancelling.

(iii) $\frac{a^2-a-6}{a^2-8a+15} = \frac{\cancel{(a-3)}^1(a+2)}{\cancel{(a-3)}_1(a-5)} = \frac{(a+2)}{(a-5)}$

Example 1.2

Simplify the following.

(i) $\frac{2}{3}\times\frac{9}{14}$ (ii) $\frac{3}{4}\div\frac{9}{16}$

(iii) $\frac{3a^2b}{2c}\times\frac{4c^3}{9ab}$ (iv) $\frac{4n^2-9}{n+1}\div\frac{2n+3}{n^2-1}$

Solution

(i) $\frac{2}{3}\times\frac{9}{14} = \frac{{}^1\cancel{2}}{{}_1\cancel{3}}\times\frac{\cancel{9}^3}{\cancel{14}_7} = \frac{1\times 3}{1\times 7} = \frac{3}{7}$

(ii) $\frac{3}{4}\div\frac{9}{16} = \frac{3}{4}\times\frac{16}{9} = \frac{4}{3}$

(iii) $\frac{3a^2b}{2c}\times\frac{4c^3}{9ab} = \frac{2ac^2}{3}$

(iv) $\frac{4n^2-9}{n+1}\div\frac{2n+3}{n^2-1} = \frac{(2n+3)(2n-3)}{n+1}\times\frac{(n+1)(n-1)}{2n+3}$

$= (2n-3)(n-1)$

Discussion point

➜ What is a common denominator?

To add or subtract fractions it is first necessary to find a *common denominator*.

Example 1.3

Simplify the following.

(i) $\frac{2}{3}+\frac{3}{4}$ (ii) $\frac{5x}{6}+\frac{x}{4}$

(iii) $\frac{2}{(x+1)}+\frac{5}{(x-1)}$ (iv) $\frac{a}{a^2-1}-\frac{2}{a+1}$

Solution

(i) $\frac{2}{3}+\frac{3}{4} = \frac{8}{12}+\frac{9}{12} = \frac{17}{12}$

(ii) $\frac{5x}{6}+\frac{x}{4} = \frac{10x}{12}+\frac{3x}{12} = \frac{13x}{12}$

The new denominator is the LCM of 6 and 4.

(iii) $$\frac{2}{(x+1)} + \frac{5}{(x-1)} = \frac{2(x-1)}{(x+1)(x-1)} + \frac{5(x+1)}{(x+1)(x-1)}$$
$$= \frac{2x - 2 + 5x + 5}{(x+1)(x-1)}$$
$$= \frac{7x+3}{(x+1)(x-1)}$$

Here you must factorise before looking for the common denominator.

(iv) $$\frac{a}{a^2-1} - \frac{2}{a+1} = \frac{a}{(a-1)(a+1)} - \frac{2}{a+1}$$
$$= \frac{a}{(a-1)(a+1)} - \frac{2(a-1)}{(a-1)(a+1)}$$
$$= \frac{a - 2a + 2}{(a-1)(a+1)}$$
$$= \frac{2-a}{(a-1)(a+1)}$$

Exercise 1.1

Simplify the following.

① (i) $\dfrac{4a^2b}{2ab^2}$ (ii) $\dfrac{6x^2y^3}{9xy^4}$ (iii) $\dfrac{4ab^3}{10a^3b}$ (iv) $\dfrac{6p^2q^3r}{3p^3q^3r^2}$

② (i) $\dfrac{2(x+3)}{4x+12}$ (ii) $\dfrac{a+2}{a^2-a-6}$ (iii) $\dfrac{3(x+y)}{x^2-y^2}$ (iv) $\dfrac{2p}{6p-2p^2}$

③ (i) $\dfrac{x^2-4x+3}{2x-6}$ (ii) $\dfrac{x^2+xy}{x^2-y^2}$ (iii) $\dfrac{3x^2+3xy}{6xy+6y^2}$ (iv) $\dfrac{9x^2-1}{9x+3}$

④ (i) $\dfrac{3a}{b^2} \times \dfrac{b^3}{6a}$ (ii) $\dfrac{xy-y^2}{y} \times \dfrac{x}{x-y}$

(iii) $\dfrac{x^2-4x+4}{x^2-2x} \times \dfrac{x-2}{x^2-4}$ (iv) $\dfrac{4p^2+12}{p-3} \times \dfrac{p^3-9p}{p^2+3}$

⑤ (i) $\dfrac{x-1}{2x} \div \dfrac{4x^2-4}{x^2}$ (ii) $\dfrac{3a^2+a-2}{2} \div \dfrac{6a^2-a-2}{8a+4}$

(iii) $\dfrac{2x-1}{x+1} \div \dfrac{2x^2-x-1}{x^2+3x+2}$ (iv) $\dfrac{3x^2-9}{x+2} \div \dfrac{x^2-6x+9}{x^2+x-2}$

⑥ (i) $\dfrac{2a}{3} + \dfrac{3a}{2}$ (ii) $\dfrac{2}{3a} + \dfrac{3}{2a}$ (iii) $\dfrac{3a}{5} - \dfrac{a}{4}$ (iv) $\dfrac{5}{3a} - \dfrac{4}{a}$

⑦ (i) $\dfrac{2}{a^2+a} + \dfrac{3}{a^2-a}$ (ii) $\dfrac{2x}{x-y} + \dfrac{2y}{y-x}$

(iii) $\dfrac{a-b}{a+b} + \dfrac{a+b}{a-b}$ (iv) $\dfrac{2p+3}{3p} + \dfrac{3p}{2p+3}$

⑧ (i) $\dfrac{p}{p^2-1} - \dfrac{1}{p+1}$ (ii) $\dfrac{2}{(m+n)} - \dfrac{1}{(m-n)}$

(iii) $\dfrac{5(2x+1)}{3x} - \dfrac{3(2x-1)}{5x}$ (iv) $\dfrac{4}{p-2} - \dfrac{3}{2p+1}$

9 (i) Simplify $\frac{5}{x-1} - \frac{3}{x-7}$

(ii) Solve the equation $\frac{5}{x-1} - \frac{3}{x-7} = 0$

(iii) What is the difference between simplifying algebraic fractions and solving an equation involving algebraic fractions?

Simplifying expressions containing square roots

In mathematics there are times when it is helpful to be able to manipulate square roots, rather than just find their values from your calculator. This ensures that you are working with the exact value, not just a rounded version.

Although calculators today will perform the numerical operations below for you, exactly the same techniques apply in algebra, as you will see in Examples 1.5 and 1.7.

Example 1.4

Simplify the following.

(i) $\sqrt{8}$ (ii) $\sqrt{6} \times \sqrt{3}$

(iii) $\sqrt{32} - \sqrt{18}$ (iv) $\sqrt{12} + \sqrt{27}$

(v) $\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right)$

Numbers like these involving at least one square root are called surds.

Solution

(i)
$$\begin{aligned}\sqrt{8} &= \sqrt{2\times2\times2}\\ &= \sqrt{2}\times\sqrt{2}\times\sqrt{2}\\ &= \left(\sqrt{2}\right)^2\times\sqrt{2}\\ &= 2\sqrt{2}\end{aligned}$$

(ii)
$$\begin{aligned}\sqrt{6}\times\sqrt{3} &= \sqrt{6\times3}\\ &= \sqrt{2\times3\times3}\\ &= \left(\sqrt{3}\right)^2\times\sqrt{2}\\ &= 3\sqrt{2}\end{aligned}$$

(iii)
$$\begin{aligned}\sqrt{32}-\sqrt{18} &= \sqrt{16\times2}-\sqrt{9\times2}\\ &= 4\sqrt{2} - 3\sqrt{2}\\ &= \sqrt{2}\end{aligned}$$

(iv)
$$\begin{aligned}\sqrt{12}+\sqrt{27} &= \sqrt{4\times3}+\sqrt{9\times3}\\ &= 2\sqrt{3}+3\sqrt{3}\\ &= 5\sqrt{3}\end{aligned}$$

(v)
$$\begin{aligned}\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right) &= 16-4\sqrt{3}+4\sqrt{3}-\left(\sqrt{3}\right)^2\\ &= 16-3\\ &= 13\end{aligned}$$

Notice that in this answer there is no square root. The expansion involves the difference of squares.

Example 1.5

Simplify the following.

(i) $\sqrt{a^3}$ (ii) $\sqrt{ab} \times \sqrt{a}$

(iii) $\sqrt{ab^2} - \sqrt{ac^2}$ (iv) $\sqrt{a^2b} - \sqrt{a^2c}$

(v) $\left(a + \sqrt{b}\right)\left(a - \sqrt{b}\right)$

Solution

(i) $\sqrt{a^3} = \sqrt{a \times a \times a}$
$= a\sqrt{a}$

(ii) $\sqrt{ab} \times \sqrt{a} = \sqrt{a^2b}$
$= a\sqrt{b}$

(iii) $\sqrt{ab^2} - \sqrt{ac^2} = b\sqrt{a} - c\sqrt{a}$
$= \sqrt{a}(b - c)$

(iv) $\sqrt{a^2b} - \sqrt{a^2c} = a\sqrt{b} - a\sqrt{c}$
$= a\left(\sqrt{b} - \sqrt{c}\right)$

(v) $\left(a + \sqrt{b}\right)\left(a - \sqrt{b}\right) = a^2 - a\sqrt{b} + a\sqrt{b} - b$
$= a^2 - b$

This answer also does not include a square root as it is a difference of squares: it is rational. When you come to Examples 1.6 and 1.7 you will see how this expansion is used in rationalising the denominator.

Notice, also, how there is less work in this example than in the previous one, since you don't have to start by looking for common factors.

In the next example, all the expressions involve fractions with a square root on the bottom line.

It is easier to work with expressions both in arithmetic and in algebra if any square roots are on the top line only. Manipulating a number to this form is called *rationalising the denominator.*

Discussion point

→ What is a rational number?

Example 1.6

Simplify the following by rationalising their denominators.

(i) $\dfrac{2}{\sqrt{3}}$ (ii) $\sqrt{\dfrac{3}{5}}$

(iii) $\sqrt{\dfrac{3}{8}}$ (iv) $\dfrac{1}{2 + \sqrt{3}}$

Solution

(i) $\dfrac{2}{\sqrt{3}} = \dfrac{2}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$
$= \dfrac{2\sqrt{3}}{\left(\sqrt{3}\right)^2}$
$= \dfrac{2\sqrt{3}}{3}$

Notice how multiplying the top and bottom by $\sqrt{3}$ results in a denominator with no square root in it; it has been **rationalised**.

(ii) $\sqrt{\dfrac{3}{5}} = \dfrac{\sqrt{3}}{\sqrt{5}}$
$= \dfrac{\sqrt{3}}{\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}}$
$= \dfrac{\sqrt{3} \times \sqrt{5}}{\left(\sqrt{5}\right)^2}$
$= \dfrac{\sqrt{15}}{5}$

(iii) $\sqrt{\frac{3}{8}} = \frac{\sqrt{3}}{\sqrt{8}}$

$= \frac{\sqrt{3}}{2\sqrt{2}}$

$= \frac{\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{3} \times \sqrt{2}}{2\left(\sqrt{2}\right)^2}$

$= \frac{\sqrt{6}}{4}$

(iv) $\frac{1}{2 + \sqrt{3}} = \frac{\left(2 - \sqrt{3}\right)}{\left(2 + \sqrt{3}\right)\left(2 - \sqrt{3}\right)}$

$= \frac{\left(2 - \sqrt{3}\right)}{4 - 3}$

$= 2 - \sqrt{3}$

Notice in part (iv) how multiplying the top and bottom by $2 - \sqrt{3}$ rationalises the denominator.

Each part of the next example is similar to the ones above but uses algebra instead of arithmetic.

Example 1.7

Simplify the following by rationalising their denominators.

(i) $\frac{a}{\sqrt{b}}$ (ii) $\sqrt{\frac{a}{b}}$ (iii) $\sqrt{\frac{a}{b^3}}$ (iv) $\frac{1}{a + \sqrt{b}}$

Solution

(i) $\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}}$

$= \frac{a\sqrt{b}}{b}$

(ii) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

$= \frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}}$

$= \frac{\sqrt{ab}}{b}$

(iii) $\sqrt{\frac{a}{b^3}} = \frac{\sqrt{a}}{\sqrt{b^3}}$

$= \frac{\sqrt{a}}{b\sqrt{b}}$

$= \frac{\sqrt{a}}{b\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}}$

$= \frac{\sqrt{ab}}{b^2}$

(iv) $\frac{1}{a + \sqrt{b}} = \frac{\left(a - \sqrt{b}\right)}{\left(a + \sqrt{b}\right)\left(a - \sqrt{b}\right)}$

$= \frac{a - \sqrt{b}}{a^2 - b}$

Look back to part (iv) of Example 1.5. The result obtained there is used here.

Example 1.8

Simplify:

(i) $\frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}$ (ii) $\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}$.

Solution

(i) Multiply the top and bottom of the fraction by $\left(\sqrt{3}-\sqrt{2}\right)$:

$$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{\left(\sqrt{3}-\sqrt{2}\right)}{\left(\sqrt{3}+\sqrt{2}\right)} \times \frac{\left(\sqrt{3}-\sqrt{2}\right)}{\left(\sqrt{3}-\sqrt{2}\right)}$$

$$= \frac{3-\sqrt{6}-\sqrt{6}+2}{3-2}$$

$$= 5-2\sqrt{6}$$

(ii) Multiply the top and bottom of the fraction by $\left(\sqrt{a}-\sqrt{b}\right)$:

$$\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{\left(\sqrt{a}-\sqrt{b}\right)}{\left(\sqrt{a}+\sqrt{b}\right)} \times \frac{\left(\sqrt{a}-\sqrt{b}\right)}{\left(\sqrt{a}-\sqrt{b}\right)}$$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{a-b}$$

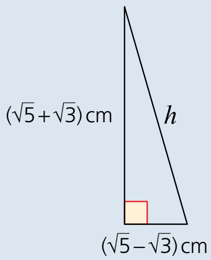
Discussion point

→ How would you rationalise the denominator of $\dfrac{1}{\sqrt{a}-\sqrt{b}}$?

Example 1.9

A right-angled triangle has shorter sides of lengths $\left(\sqrt{5}+\sqrt{3}\right)$ cm and $\left(\sqrt{5}-\sqrt{3}\right)$ cm. Find the length of the hypotenuse.

Solution



Let the length of the hypotenuse be h.

Using Pythagoras' theorem

$$h^2 = \left(\sqrt{5}+\sqrt{3}\right)^2 + \left(\sqrt{5}-\sqrt{3}\right)^2$$

$$= \left(5+2\sqrt{15}+3\right) + \left(5-2\sqrt{15}+3\right)$$

$$= 8+2\sqrt{15}+8-2\sqrt{15}$$

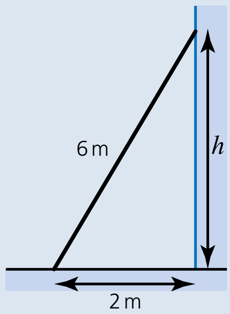
$$= 16$$

The length of the hypotenuse is 4 cm.

Example 1.10

A ladder of length 6 m is placed 2 m from a vertical wall at the side of a house. How far up the wall does the ladder reach? Give your answer as a surd in its simplest form.

Solution



Using Pythagoras' theorem

$$6^2 = 2^2 + h^2$$

$$\Rightarrow 36 = 4 + h^2$$

$$\Rightarrow 32 = h^2$$

$$\Rightarrow h = \sqrt{32}$$

h must be positive.

$$= 4\sqrt{2}$$

The ladder reaches $4\sqrt{2}$ metres up the wall.

Notice that $\sqrt{32}$ is not in its simplest form. It is always good practice to simplify surds into their simplest form, in this case $4\sqrt{2}$. In this example the question actually told you to do this.

Exercise 1.2

Do not use a calculator for this exercise.

① Write each of the following in its simplest form.

(i) $\sqrt{12}$ (ii) $\sqrt{75}$ (iii) $\sqrt{300}$

(iv) $3\sqrt{5} + 6\sqrt{5}$ (v) $\sqrt{48} + \sqrt{27}$ (vi) $3\sqrt{45} - 2\sqrt{20}$

② Write each of the following in its simplest form.

(i) $\sqrt{a^3}$ (ii) $\sqrt{27a^2}$ (iii) $\sqrt{16a}$

(iv) $\sqrt{a^2b} + \sqrt{ab^2}$ (v) $\sqrt{3a^2} + \sqrt{27a^2}$ (vi) $\sqrt{3ab} + \sqrt{27ab}$

③ Express each of the following as the square root of a single number.

(i) $3\sqrt{6}$ (ii) $5\sqrt{5}$

(iii) $12\sqrt{3}$ (iv) $10\sqrt{17}$

④ Simplify the following, leaving all square roots in the numerator.

(i) $\sqrt{\dfrac{25}{49}}$ (ii) $\sqrt{\dfrac{24}{9}}$

(iii) $\sqrt{\dfrac{12}{15}}$ (iv) $\sqrt{\dfrac{6}{121}}$

⑤ Simplify the following leaving all square roots in the numerator.

(i) $\sqrt{\dfrac{a^2}{b^2}}$ (ii) $\sqrt{\dfrac{12a}{9b^2}}$

(iii) $\sqrt{\dfrac{a^3}{8b^2}}$ (iv) $\sqrt{\dfrac{3ab^2}{12a^2b}}$

⑥ Simplify the following by collecting like terms.

(i) $\left(3 + \sqrt{2}\right) + \left(5 + 4\sqrt{2}\right)$ (ii) $4\left(\sqrt{3} - 1\right) + 4\left(\sqrt{3} + 1\right)$

⑦ Expand and simplify:

(i) $\left(\sqrt{3} + 2\right)\left(\sqrt{3} - 2\right)$ (ii) $\sqrt{3}\left(5 - \sqrt{3}\right)$

(iii) $\left(4 + \sqrt{2}\right)^2$ (iv) $\left(\sqrt{6} - \sqrt{3}\right)\left(\sqrt{6} - \sqrt{3}\right)$.

⑧ Expand and simplify:

(i) $\left(a + 2\sqrt{b}\right)\left(a - 2\sqrt{b}\right)$ (ii) $\left(3\sqrt{a} - 4\sqrt{b}\right)^2$

(iii) $\left(2\sqrt{a} + 3\sqrt{b}\right)^2$ (iv) $\sqrt{ab}\left(\sqrt{a} - \sqrt{b}\right)$.

⑨ Rationalise the denominators, giving each answer in its simplest form.

(i) $\dfrac{1}{\sqrt{6}}$ (ii) $\dfrac{12}{\sqrt{3}}$ (iii) $\dfrac{\sqrt{6}}{2\sqrt{2}}$

(iv) $\dfrac{1}{\left(\sqrt{5} - \sqrt{2}\right)}$ (v) $\dfrac{3 + \sqrt{2}}{4 - \sqrt{2}}$ (vi) $\dfrac{3 - \sqrt{5}}{5 + \sqrt{5}}$

⑩ Rationalise the denominators, giving each answer in its simplest form.

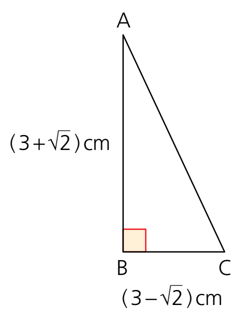
(i) $\dfrac{1}{\sqrt{ab}}$ (ii) $\dfrac{\sqrt{ab^2}}{\sqrt{a^2b}}$ (iii) $\dfrac{\sqrt{ab}}{3\sqrt{b^3}}$

(iv) $\dfrac{1}{\sqrt{a} - \sqrt{b}}$ (v) $\dfrac{2 + \sqrt{a}}{2 - \sqrt{a}}$ (vi) $\dfrac{2\sqrt{a} - \sqrt{b}}{\sqrt{a} + 2\sqrt{b}}$

⑪ Write the following in the form $a + b\sqrt{c}$ where c is an integer and a and b are rational numbers.

(i) $\dfrac{1+\sqrt{2}}{3-\sqrt{2}}$ (ii) $\dfrac{3\sqrt{5}}{3+\sqrt{5}}$ (iii) $\dfrac{2\sqrt{6}}{\sqrt{6}-2}$

PS ⑫ Find the length of AC of the triangle shown below.



⑬ A square has sides of length x cm and diagonals of length 12 cm. Use Pythagoras' theorem to find the exact value of x and find the area of the square.

⑭ An equilateral triangle has sides of length $\sqrt{3}$ cm.

Find:

(i) the height of the triangle

(ii) the area of the triangle in its simplest surd form.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to:

➤ simplify expressions involving algebraic fractions and square roots.

KEY POINTS

1 When simplifying an algebraic fraction involving multiplication or division you must factorise first and then you can cancel by a common factor.

2 When simplifying an algebraic fraction involving addition or subtraction you first need to find a common denominator. This may not just be the product of existing denominators.

3 When simplifying expressions involving square roots you should:
- make the number under the square root sign as small as possible
- rationalise the denominator.

FUTURE USES

The ability to manipulate algebraic expressions pervades the whole of Pure Mathematics at this level and beyond.