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Introduction

One of the main philosophies behind the Making Sense of Maths materials is the delaying of formal procedures. Rules and algorithms emerge from students' own ideas and stay connected to those ideas so that students can make sense of them. Nowhere is this more evident than in the work on area and volume.

Our experience from teaching, observing, and analysing students' work is that too often area and volume are quickly reduced to numerical procedures that, certainly in the minds of the students, bear no resemblance to anything to do with 'filling', 'shape' or 'space'. When these students are then faced with area and volume problems, their only recourse is to try and remember the procedures. Particularly for students who find mathematics difficult, this is rarely a recipe for success.

As part of the trials for these materials, students were asked to find the area of a trapezium. The students concerned had not at this stage met the formula for this shape. Some of the students had been working with our materials ('project students') and some had not ('control students'). The work of over 400 students was analysed.

When faced with the question:

Find the area of the shape shown below. Show carefully how you worked it out.

the overwhelming majority of control students simply decided to adopt a numerical procedure, usually involving a combination of multiplying and dividing. Some typical responses from these students are reproduced here:

A control student attempting two different multiplication approaches, with no attempt to make sense of the problem from an area perspective.

A control student making no sense of the problem, but simply attempting to 'do something with the numbers'.

A control student attempting two different multiplication approaches, with no attempt to make sense of the problem from an area perspective.
Perhaps most telling of all was the student who didn’t actually attempt the question but simply wrote, ‘I can’t remember how to do area like this, but I do remember it’s got something to do with timesing’!

When the same question was given to project students, a very different picture emerged. Of course, there were still plenty of these students who multiplied numbers together, but far more adopted other strategies. These ranged from drawing and counting squares, to splitting into triangles and rectangles, to ‘reallocating’ a triangle. Altogether, over 60% of project students adopted such strategies, though of course not all got the final answer correct.

This second example is interesting for a number of reasons.

Teachers have reported that students sometimes feel that drawing squares (or drawing anything!) is ‘babyish’ and not ‘proper’ maths, and are resistant to this. Yet ‘fitting in’ squares and cubes is the basis of all work on area and volume, and it is important that teachers resist students’ (and their own!) desires for formulas and ‘quick ways’ to arrive at answers. It was not unusual in project classes, even in revision lessons, to hear teachers asking questions such as, ‘Why have you multiplied the numbers together?’, ‘How does your calculation count the squares?’, ‘How are you stacking the cubes?’ etc.

We obviously want students to have procedures for solving problems, but not at the expense of understanding. We want the procedures to stay connected to what students understand so that, if they forget, they have something to fall back on.

In trials, teachers have come to see their role as one that values student intuitions and supports mathematical development whilst keeping hold of the meaning behind the symbols and procedures. They have found that they sometimes spend more time on topics but do not need to revisit topics as much as in the past. Also, when they do, they do not have to start ‘from the beginning’ as much.

We hope that this approach ‘makes sense’ to you, and that your students experience success through working in this way.
Introduction

We've designed the contexts in this chapter so that students use their own knowledge and intuition to tackle problems about area and perimeter. The strategies associated with these contexts are:

- Re-allotting an area to compare it with another, e.g. ‘A green and pleasant land?’ on page 1 of the Student’s Book.
- Using a mediating quantity such as the price of a tile to work out a price for other tiles, e.g. ‘Pricing tiles’ on page 3 of the Student’s Book.
- Seeing a triangle as half a rectangle and the implication this has for counting squares, e.g. ‘Stained glass’ on page 4 of the Student’s Book.
- Adopting smart ways of counting for a large array, e.g. ‘Cleaning windows’ on page 5 of the Student’s Book.
- Visualising the squares inside an ‘empty’ shape, e.g. ‘Touching the numbers: the rectangle’ on page 6 of the Student’s Book.

We have deliberately avoided showing students the traditional formulas for finding areas. For example, they approach finding the area of a trapezium as a problem to be solved by visualising and counting squares in a variety of ways, not by using a set formula. However, it is likely that students will evolve their own methods that look very similar to the traditional formulas. The difference is that these will be based on their own sense of what it means to count the number of squares inside a shape.

Sub sections

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<th>Questions</th>
<th>Intended emphasis</th>
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<tr>
<td>Q1–8</td>
<td>i) Comparing areas by superimposing one area on top of another using tracing paper. ii) The strategy of quantifying areas using a unit square is available in the costing tiles problems, but this is only one of a number of strategies offered by the students.</td>
</tr>
<tr>
<td>Workbook exercises 1.1, 1.2, 1.3, 1.4</td>
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<tr>
<td>Q9–11</td>
<td>i) Seeing an area as a fraction of a known area (using the quantity of price as a mediating quantity for area). ii) Re-allotting parts of shapes to create shapes where the area can more easily be seen.</td>
</tr>
<tr>
<td>Workbook exercises 1.5, 1.6, 1.7</td>
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<tr>
<td>Q12–19</td>
<td>i) Visualising the squares contained within a shape even when it is drawn on plain paper. Tasks involve students actually drawing in the rows of squares and developing methods for smart (fast) counting. ii) Seeing the distance across a shape in various places, so as to encourage visualisation of the squares contained within. iii) The strategy of re-allotting is revisited as a possible strategy for finding the area of a trapezium.</td>
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</table>

Resources

Workbook exercises 1.1 to 1.7 in Workbook
Tracing paper
String
Grid overlay
Rulers
Lesson 1: A green and pleasant land? The delivery driver. Pricing tiles

Possible lesson outline

Objective: To develop strategies for comparing the size of regions

<table>
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<tr>
<th>Lesson phase</th>
<th>Activity</th>
<th>Comments</th>
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</thead>
<tbody>
<tr>
<td>Whole class – Introduction Q1–5</td>
<td>Discussion of how to fit one country in another using the maps projected at the front of the classroom.</td>
<td>Look out for students gesturing the idea of fitting one country/region into another. A common misconception may be to think land size and population are proportional. For actual population figures see Solutions and strategies on page 4.</td>
</tr>
<tr>
<td>Individual/Pairs Q6–7</td>
<td>Students identify regions of the same size, either by eye or by using tracing paper to re-allot parts of the region.</td>
<td>Some students may believe that perimeter has an effect when comparing the region sizes.</td>
</tr>
<tr>
<td>Individual and whole class Q8</td>
<td>Students individually decide on prices for the tiles (using tracing paper if required) before reviewing their ideas as a class.</td>
<td>There will be several different strategies around – see Solutions and strategies on page 6. The parallelogram is tricky; many students believe that it is a diamond/rhombus.</td>
</tr>
<tr>
<td>Workbook exercise 1.1 Q1</td>
<td>Students decide on prices for the other tiles. Some questions could be used for homework.</td>
<td>There will be various strategies: re-allotting parts; fractions of a large tile; fitting in small tiles. See Solutions and strategies on page 7.</td>
</tr>
</tbody>
</table>
A green and pleasant land?

Using the map above, estimate the answer to the following questions. Describe how you found each answer.

1. a) How many Northern Irelands will fit into Scotland?
   b) How many Northern Irelands will fit into England?
   c) How many Scotlands will fit into England?

2. In 2006 the population of Wales was around 3 million people. Write down what you think were the populations of:
   a) Northern Ireland
   b) Scotland
   c) England

3. How do your estimates compare with the actual population figures?

4. How can you explain any differences?

A green and pleasant land? Solutions and strategies

1. a) roughly 6
   b) roughly 13
   c) roughly 2

Some students may trace over Northern Ireland and place it over Scotland to see how many times it will fit. Others may vaguely make the shape of N Ireland with their fingertips and again see how many fit in the Scotland outline. Some may ask for string and attempt a method with it, which could suggest confusion with the relationship between size and edge length.

A grid overlay provides an alternative strategy: counting the relative number of squares that fit into each country.

It is worth making the students aware of the variety of materials on offer, but allowing each to make their own choice about which to use and how to use them.

Continued
2 Some students may work on a ‘population is proportional to area’ model, in which case they are likely to estimate figures similar to:

a) N.I. – half as much 1.5 million
b) Scot. – $2\frac{1}{2}$ to 3 times as much 8 million
c) Eng. – roughly 6 times as much 18 million

Some students may recognise that England is much more densely populated than Wales and therefore find it harder to estimate population figures.

It is worth investing time in debates about the context, as this helps the students to develop a sense-making approach to tackling the problems.

3 The actual population figures for 2006 were:

<table>
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<th>Country</th>
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<tr>
<td>Wales</td>
<td>3 million</td>
</tr>
<tr>
<td>Northern Ireland</td>
<td>1.7 million</td>
</tr>
<tr>
<td>Scotland</td>
<td>5.1 million</td>
</tr>
<tr>
<td>England</td>
<td>50.4 million</td>
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</table>

Students who used a linear model to estimate their figures will hopefully be surprised by the vast difference, particularly in the case of England. Some students may conclude that Northern Ireland has a similar population density to Wales.

4 See question 2 and question 3 above for the sorts of explanations students may offer.
Chapter 1: Looking at the size of shapes

The delivery driver

Digital Deliveries Limited is a company that deals in electronic equipment in the UK. The map above shows the delivery regions for this company. In which type of region would you prefer to be a van driver? Explain.

Matthew covers the Yorkshire and Lincolnshire region. He claims that he has a bigger region to cover than Morgan, who drives in the South West region. Which region do you think is bigger? Explain your reasoning.

See if you can identify other pairs of regions that are roughly the same size.

5 Students may give a variety of preferences such as ‘The pink one because it’s by the seaside’ or ‘The light yellow one because it’s smaller’, etc. Some students recognise that places like London are more built up, so you would be driving in more traffic. Others struggle to identify the location of various places on the map.

6 Matthew’s region is slightly bigger. Students may trace one region, rotate it and place it over the other, and then compare the non-overlapping areas. Some students may cut off and re-position parts of their tracings. Others may use a grid overlay.

E.g. Matthew’s region has been superimposed on Morgan’s region:

7 Light blue and light pink; mid-green and light green; bright pink and purple. Students may make suggestions as to comparable regions by eye. They might also use a variety of strategies as described in question 6 above to help verify their claims.
Pricing tiles: Solutions and strategies
Workbook exercise 1.1

The answers to questions 1–4 follow. The lists correspond to the answer boxes surrounding the pictures in the Workbook; starting in the top left-hand corner and working clockwise in each case.

1. £1.80 \div 9 = 20p (roughly 9 small tiles make up one big). £1.80 (the sand coloured tile is roughly the same size as the white tile).

Some students may argue that it could be a different price because it is a different colour.

£1.20 (if you re-allot the left hand triangle as shown below, then the resulting shape is roughly two thirds of a whole tile, or 6 little tiles).

80p (if you re-allot the big triangle as shown below, it would make up roughly 4 small tiles).

Many students argue that this triangular tile is half the whole tile, so should cost 90p. (In reality it probably was made by cutting the whole tile along the diagonal, so you can argue it either way. The inconsistency comes from the spacing between tiles.)

In this question students naturally employ a variety of strategies but direct use of any area formulae is not usually one of them. They may see how many of the smallest square tile fit into the other tiles. They may re-allot parts of tiles.

Some may find the use of tracing paper helpful.

In this question we see price being used as a ‘mediating quantity’ for area. This is also the case in the remaining questions that follow.

Here is a design for an arrangement of tiles that could be used in the home or garden:

The price of a standard-sized tile as shown by the arrow is £1.80.

Work out fair prices for the other tiles in this design.

There is a copy of this picture in Workbook exercise 1.1 (on page 1 of your workbook) which you can use to answer this question.

Discuss your strategies with your classmates.

Turn to pages 1–3 in your workbook and do the other questions in Workbook exercise 1.1.
2 £3.60 (roughly 2 square tiles).
   90p (roughly half a square).
   £7.20 (4 of the priced tiles or 2 of the £3.60 tiles).
   The answers are much more clear-cut, so students are unlikely to require as much
discussion.

3 90p (It is roughly half a square tile cut diagonally).
   30p (You can fit nearly 6 of these in the £1.80 tile).
   £1.50 (If cut in half this fits into the priced tile twice with a thin strip left. The thin
   strip is roughly equal to 1 small square tile. £1.80 – 30p = £1.50).
   It is extremely difficult to be precise with these answers, since neither the long
   white tile nor the small square tile fit exactly into the £1.80 tile. Again, students
   may find tracing paper helpful as a way of obtaining a reasonable estimate.

4 40p (9 of these make up the priced tile).
   £1.20 (roughly three of the small square tiles).
   £1.00 (equal to 2\(\frac{1}{2}\) small square tiles).
   60p (equal to 1\(\frac{1}{2}\) small square tiles).
   20p (half a small square tile).
   £1.80 (half the priced tile).
   Between £1.80 and £2.40 (the way the picture is shown it looks like just over half
   the size of the priced tile, but less than 6 of the small square tiles).
   It is easier to give some exact answers due to the geometry of the design. It is
   likely that some of the students will use area by subtraction methods for, say, the
   £3.40 tile.
Lesson 2: Stained glass

Possible lesson outline

**Objective:** To work out the cost of stained glass pieces in relation to a whole sheet of glass

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<td>Review of homework if appropriate.</td>
<td>Some students struggle to get their head around what the numbers above the lengths of the rectangles mean.</td>
</tr>
<tr>
<td>Q9, 10 and 11</td>
<td>Discussion of the context of stained glass.</td>
<td></td>
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<td></td>
<td>Hear some ideas about how they intend to approach Workbook exercise 1.2.</td>
<td></td>
</tr>
<tr>
<td>Individual</td>
<td>Students decide on the prices of glass, employing a variety of strategies.</td>
<td>See Solutions and strategies on page 9.</td>
</tr>
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<td>Workbook exercise 1.2</td>
<td>(Working out the price of a unit square is not usually one of them.)</td>
<td></td>
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<td>Whole class review</td>
<td>Students demonstrate and listen to each other’s approaches to Workbook exercise 1.2.</td>
<td>See Solutions and strategies on pages 9 and 10. Once the students have re-allotted or seen the piece as a fraction of a rectangle, they will tend to count the squares inside rather than multiplying lengths.</td>
</tr>
<tr>
<td>Workbook exercise 1.2</td>
<td>Students work through Workbook exercise 1.3, employing a variety of strategies.</td>
<td></td>
</tr>
<tr>
<td>Individual</td>
<td>Students demonstrate and listen to each other’s approaches to Workbook exercise 1.3.</td>
<td></td>
</tr>
<tr>
<td>Workbook exercise 1.3</td>
<td>Students demonstrate and listen to each other’s approaches to Workbook exercise 1.3.</td>
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<td>Homework</td>
<td>Set up homework: Workbook exercise 1.4.</td>
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Stained glass windows have been popular in churches and houses since the 11th century. Some designs from the 19th and 20th centuries are still used today.

a) When was the house or flat that you live in built? Does it have any stained glass?
b) What about your school building?
c) Where was the last place you saw stained glass?

Look at the pictures below. Think about any stained glass windows you have seen. What sorts of shapes are popular for stained glass windows? Sketch and name some examples of shapes used in stained glass windows.

A company uses rectangular sheets of coloured glass. They cut pieces from these to make a window.

Each rectangular sheet measures 6 inches by 4 inches and costs £18.

Work out the cost of each piece of coloured glass.

a) b) c)

Now try Workbook exercises 1.2, 1.3 and 1.4 (pages 4–6) in your workbook.
Stained glass windows have been popular in churches and houses since the 11th century. Some designs from the 19th and 20th centuries are still used today.

a) When was the house or flat that you live in built? Does it have any stained glass?

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A company uses rectangular sheets of coloured glass. They cut pieces from these to make a window.

Each rectangular sheet measures 6 inches by 4 inches and costs £18.

Work out the cost of each piece of coloured glass.

a) £6

b) £12

c) £4.50

d) £3

e) £6

f) £6 (can be re-allotted to make piece a or seen as \(\frac{1}{2}\) of piece b)

g) £6 (can be re-allotted to make a \(\frac{1}{2}\) sized strip down the middle. Or subtract the white areas.)

h) £9

i) £9

j) £13.50 (£18 subtract \(\frac{1}{2}\) of shape a and \(\frac{1}{2}\) of shape a)

k) £9 (\(\frac{1}{2}\) of shape a) added to \(\frac{1}{2}\) of shape b)

l) £12 (re-allotting can give shape b)

Workout exercise 1.2

Students use a variety of informal strategies to answer these questions which may involve:

- Drawing on the diagram to find out what fraction the piece is of the whole sheet.
- Seeing one piece made up from a combination of other pieces.
- Re-allotting parts of a piece.

a) £6 (\(\frac{1}{2}\) of £18)

b) £12 (\(\frac{2}{3}\) of £18)

c) £4.50 (\(\frac{1}{2}\) of a half)

d) £3 (\(\frac{1}{2}\) of piece a)

e) £6 (\(\frac{1}{2}\) of piece)

f) £6 (can be re-allotted to make piece a or seen as \(\frac{1}{2}\) of piece b)

g) £6 (can be re-allotted to make a \(\frac{1}{2}\) sized strip down the middle. Or subtract the white areas.)

h) £9 (\(\frac{1}{2}\) of £18)

i) £9 (\(\frac{1}{2}\) of £18)

j) £13.50 (£18 subtract \(\frac{1}{2}\) of shape a and \(\frac{1}{2}\) of shape a)

k) £9 (\(\frac{1}{2}\) of shape a) added to \(\frac{1}{2}\) of shape b)

l) £12 (re-allotting can give shape b)

Workout exercise 1.3

a) 4 square units

b) 6 square units

c) 6 + 4 = 10 square units

d) 3 + 4\(\frac{1}{2}\) = 7\(\frac{1}{2}\) square units

e) 7\(\frac{1}{2}\) square units

f) 7\(\frac{1}{2}\) + 3 = 10\(\frac{1}{2}\) square units

Continued
Chapter 1: Looking at the size of shapes

A similar variety of strategies will be used as in question 11 and Workbook exercise 1.2, for example:

- Horizontally cuts the shape and re-allots to make a rectangle that fills the bottom (or top) half of the original rectangle.

- Recognises that some shapes can be made from combining other shapes together.

Workbook exercise 1.4

The most commonly used strategies are:

- Re-allot part of the parallelogram to create a rectangle. Then the squares inside can easily be counted.
- Surround the parallelogram by a rectangle and subtract the areas of the 2 right angled triangles formed at either side.

- 5 rows of 6 = 30 square units
- 3 rows of 9 = 27 square units
- 6 rows of 5 = 30 square units
- (4 rows of 7) + (2 of 8) = 32 square units
- 2 rows of 6 = 12 square units
- 4 rows of 6 = 24 square units
- 4 rows of 6 = 24 square units
- 4 rows of 6 = 24 square units (this will require more than one re-allotment in order to create a rectangle.)

Most of these involve re-allotting part of the shape to give a rectangle so that the squares can easily be counted.
Lesson 3: Cleaning windows

Possible lesson outline

**Objective:** To develop strategies for ‘fast counting’ and to start to visualise the squares inside a rectangle

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<td>Whole class – Introduction Q12</td>
<td>Review of homework if appropriate. Students individually count the number of windows before reviewing their strategies as a class.</td>
<td>Strategies will vary – see Solutions and strategies on page 12.</td>
</tr>
<tr>
<td>Individual/Pairs Workbook exercise 1.5</td>
<td>Students individually count the number of windows. They could compare their answers and strategies with their neighbours after each part.</td>
<td>Students may develop more sophisticated strategies as they move through this task. They may begin to recognise that some strategies are more efficient than others.</td>
</tr>
<tr>
<td>Whole class Class activity 1</td>
<td>Insist that the student at the board is very precise and runs their finger <em>all</em> the way along the distance of 15 in the rectangle. The second distance of 15 is identified quickly. There are likely to be different ideas about where they see a third distance of 15 and it is worth exposing these. ‘Where can you see the 120 squares...?’ – it is worth the class seeing a student at the board attempting to draw in the squares. Trying to fit 15 squares across is not a trivial task and time should be spent on drawing in some rows of squares and touch-counting them before moving on to Workbook exercise 1.6.</td>
<td>Students are initially slightly puzzled by these questions but soon get into it. It emerges that they are not clear initially that the distance of 15 can be seen in many, many places. Some believe the diagonal is 15. Others indicate the vertical to be 15. At later stages many argue that there could be an infinite amount of distances that are 15; others think that there are only 9 (or 8) places, which become visible when you start to mark in the squares.</td>
</tr>
<tr>
<td>Individual Workbook exercise 1.6 Q1 and 2 Homework</td>
<td>Students individually find the area and carefully draw (fit in) the squares row by row. Some parts could be set for homework.</td>
<td>Some students will have their own strategies for finding the areas. For them the drawing in of squares is a check that their method works (or not). Other students will only be able to find the area after they have drawn in the squares, which is also fine.</td>
</tr>
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</table>

**Mathematical focus**

Working on the relationship between repeated addition and multiplication
Cleaning windows: Solutions and strategies

12 Inevitably answers will vary and it is not always clear in the photo what counts as a window. The photos are deliberately vague (with windows covered up etc.) in order to promote a need to count efficiently, i.e. there is an implied need to count in rows or ‘groups of’ instead of touch-counting every single window. This is an opportunity to revisit multiplication, and even long multiplication, with an emphasis on ‘seeing’ the multiplications within the picture.

Workbook exercise 1.5
See comments above.

Cleaning windows

Chris and Alex work for a window cleaning firm. They specialise in cleaning high-rise buildings. They charge by the pane of glass.

How many windows can you see in the hotel building pictured below? Compare your method of counting with your neighbour’s.

Workbook exercise 1.6

Touching the numbers: The rectangle

Look at the rectangle above.

Where do you see a distance of 15 in this picture? Run your finger along exactly where you see the distance of 15.

Where else can you see a distance of 15? Again, run your finger along.

How many places can you see a distance of 15? Run your finger along these too.

Now look at the 8.

Where do you see a distance of 8 in this picture?

Where else do you see a distance of 8?

How many places can you see a distance of 8? Run your finger along all of these.

What is the area of this rectangle?

Where can you see the 120 squares in this picture?
Touching the numbers: The rectangle:
Solutions and strategies

These activities are best run as a whole class session with students coming out to the front and indicating where they can see the required distances. It is important that they are precise when outlining the distance and run their finger along the exact length from start to finish. There can be an initial reluctance on the part of the student, as if you are asking a stupid or obvious question. Most think it is very obvious that the distance is 15 at the top and bottom of this rectangle, but ‘seeing’ 15 in lots of other places can be a startling realisation for some. Some students will come out and show another distance of 15 by running their finger along the diagonal (an interesting one to try to explain why this is not in fact 15).

In response to ‘What is the area of this rectangle?’ there is likely to be a range of answers including 120, 23 and 46. Students will justify these with explanations like ‘I timesed them together’, ‘I added the numbers’, or some sort of perimeter routine.

Prompts such as ‘120 what? 23 what?’ are useful in order to obtain the unit of squares (no need for more precision than that). The word ‘squares’ is important as it is required for the follow-up question of ‘Where are the 120 squares?’ or ‘Where are the 23 squares?’ etc.

This activity is designed to encourage the students to make the link back from their formal algorithmic method for finding an area to the context of the squares and counting them.

Workbook exercise 1.6

1 a) 15 columns of 8 squares drawn in
b) and c) Students are likely to have counted either in rows going across, or in columns going down, e.g:
   1 counted 15 across and there were 8 lots of 15.
   1 counted down in eights... and got 8, 16, 24, 32 and so on up to 120.
   The language ‘lots of’ is important and worth looking for in the students’ responses.

2 a) Area = 24 + 56 = 80 squares
   b) Area = 45 + 12 = 57 squares
   c) Area = \( \frac{1}{2} \) of 50 = 25 squares
   d) Area = \( \frac{1}{2} \) of 77 = 38 \( \frac{1}{2} \) squares
   e) Area = 20 + 70 = 90 squares
   f) Area = 66 squares

Student descriptions for part iii) will vary. Encourage students to use the words ‘lots of’ or ‘rows of’ and to look for similarities and subtle differences between their own counting system and that of their neighbour.
Lesson 4: Touching the numbers: The rectangle.
Touching the numbers: The trapezium

Possible lesson outline

Objective: To develop an awareness of distances in an ‘empty’ shape

<table>
<thead>
<tr>
<th>Lesson phase</th>
<th>Activity</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole class – Introduction</td>
<td>Review of homework if appropriate. Discussion of how you might find the</td>
<td>Look out for the strategy of running a finger along the sides of the shape. This is a natural</td>
</tr>
<tr>
<td>Workbook exercise 1.6</td>
<td>perimeter of one of the shapes from Workbook exercise 1.6.</td>
<td>follow-on from Class activity 1 in the last lesson.</td>
</tr>
<tr>
<td>Q3 and 4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Individual</td>
<td>Students work on finding the perimeters. Students compare answers with</td>
<td>The misconception that the diagonal is the same length as the horizontal will still be around</td>
</tr>
<tr>
<td>Workbook exercise 1.6</td>
<td>their neighbours and justify the strategies they used to estimate the</td>
<td>for some students.</td>
</tr>
<tr>
<td>Q3 and 4</td>
<td>diagonal lengths.</td>
<td></td>
</tr>
<tr>
<td>Whole class</td>
<td>Students demonstrate to the class where they can see the distances in</td>
<td>Seeing the distance of 11 on the trapezium initially proves quite difficult.</td>
</tr>
<tr>
<td>Class activity 2</td>
<td>the trapezium.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>This should be run in the same way as Class activity 1 (see previous</td>
<td></td>
</tr>
<tr>
<td></td>
<td>lesson).</td>
<td></td>
</tr>
<tr>
<td>Individual</td>
<td>Students attempt to find the area of the trapezium. They may wish to</td>
<td>Students may start to recognise that not knowing where the 9 is in relation to the 20 is a</td>
</tr>
<tr>
<td>Q13</td>
<td>use tracing paper, plain paper or squared paper. They need to be able to</td>
<td>problem.</td>
</tr>
<tr>
<td></td>
<td>show someone how they know their answer is correct.</td>
<td></td>
</tr>
<tr>
<td>Whole class</td>
<td>Students at the board point to the diagrams and explain to the class</td>
<td>Some students may relate quickly to Tom and Sandra’s methods as they may well concur with</td>
</tr>
<tr>
<td>Q14, 15 and 16</td>
<td>what they think Tom and Sandra have done.</td>
<td>the strategies they used themselves in Q13. Some students may start to recognise that</td>
</tr>
<tr>
<td>Homework</td>
<td>Q17 could be used as homework.</td>
<td>the area of a trapezium does not change in the case of Q17.</td>
</tr>
</tbody>
</table>

Mathematical focus
Finding perimeters.
Developing strategies for finding the area of a trapezium
Workbook exercise 1.6

3  
   a) $P = 42$ unit lengths  
   b) $P = 36$ unit lengths  
   c) $P = 5 + 10 + \text{roughly } 12 = 27$ unit lengths (many students will incorrectly assume the diagonal length is 10. An estimation of above 10 is intended rather than any exact calculation of this diagonal length).  
   d) $P = 11 + \text{roughly } 9 + \text{roughly } 10 = 30$ unit lengths (using estimation by eye).  
   e) $P = 10 + 11 + 5 + 4 + \text{roughly } 8 = 38$ unit lengths (using estimation by eye).  
   f) $P = (2 \times 11) + (2 \times \text{roughly } 7) = 36$ unit lengths (using estimation by eye).

4  
   a) exact  
   b) exact  
   c) approximate  
   d) approximate  
   e) approximate  
   f) approximate

Touching the numbers: The trapezium: Solutions and strategies

Again, this is best run as a whole class activity. It is important that the students working at the board are asked to be extremely precise when indicating the distance requested. Repetition of 'seeing' the distance is also important as a way of developing this visualisation.

Students may well have difficulty trying to find the area of the trapezium. The problem does not need to be resolved at this stage. Recognition that they can see how many squares will fit in the rectangle part, but are not sure about the triangles, would indicate progress. Some students may suggest that not knowing where the 9 is in relation to the 20 is a possible problem. Any indication that the students are thinking in terms of squares rather than 'number crunching' is progress.
14 Descriptions will vary. Tom’s method may be very similar to the way some of the students tackled this problem. It can be helpful if students record their exact descriptions on the board and then these can be studied and scrutinised by the rest of the class, to see if people can follow them.

15 Descriptions will vary. Some students may struggle to see where the shaded triangle has come from and why the base is 11.

16 Opinions will vary. Some students may be able to argue that $20 \times 8$ (or 8 rows of 20) is a big rectangle, bigger than the trapezium, so Tom’s method can’t be right.

16 Below is Tom’s attempt to answer this question. Describe what you think Tom has done.

15 Below is Sandra’s attempt to answer the question. Describe what you think Sandra has done.

16 Whose method (if any) do you agree with?
17 a) Look at these trapeziums. What features do they have in common?

b) Find the area of each one.

17 a) Students may notice a number of common features: each trapezium has a base of 7, a height of 4, a top length of 3 and an area of 20 squares.

b) Some strategies for finding the areas are described below:

Splits the trapezium into a rectangle and two triangles.

Uses previous methods for finding areas of the three shapes, i.e. sees the right-hand triangle as half of a 3 by 4 rectangle, or re-allots the top part of the RH triangle to make a 3 by 2 rectangle.

Some students may start to recognise the potential for Sandra’s method here, i.e. if you put the LH and RH triangles together then you get a triangle that is half the size of a 4 by 4 rectangle.

Students may split the last trapezium in a variety of ways: a 2 by 4 rectangle and two triangles; two triangles by drawing in a diagonal; or a parallelogram of base length 3 and a triangle.
Lesson 5: Playground surfaces. Summary

**Possible lesson outline**

**Objective:** To consider and justify a variety of ways of costing a playground surface

<table>
<thead>
<tr>
<th>Lesson Phase</th>
<th>Activity</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole class – introduction Q18</td>
<td>Discussion of the context of playground surfaces.</td>
<td></td>
</tr>
<tr>
<td>Individual Q19</td>
<td>Students trace the playground and experiment with ways of ‘fitting in’ the 1 m by 1 m square in order to cost the surface.</td>
<td>Encourage students to make more than one attempt at this. Suggest scissors to help those who have identified that it is the left- and right-hand triangles that are a problem.</td>
</tr>
<tr>
<td>Whole class review</td>
<td>Students report on and demonstrate to the class how they went about Q19. Some conflict as students disagree with others and are forced to justify their thinking.</td>
<td>Encourage a variety of methods – see Solutions and strategies on page 20.</td>
</tr>
<tr>
<td>Individual</td>
<td>Students individually work out the prices of various shaped play areas.</td>
<td>Most areas are trapezium-shaped, though students are likely to treat each one as a problem in its own right. An estimated price is what is intended for Q4, the circular shape, at this stage.</td>
</tr>
<tr>
<td>Workbook exercise 1.7</td>
<td>Some parts could be set as homework.</td>
<td></td>
</tr>
</tbody>
</table>

**Mathematical focus**

Finding the area of a trapezium and other 2-D shapes
Playground surfaces

18 Look at the picture below.
   a) What do you think the surface of the playground is made of?
   b) Have you ever fallen over in a playground? What type of surface did you land on?

Playground surfaces: Solutions and strategies

18 The purpose of this question is for students to embrace the context of surfaces used in playgrounds. They are likely to have vivid memories from their childhood and even recent experiences of hanging round in parks. Some students will want to offer detailed accounts of the times they fell in a playground.
Strategies will vary:

Some students will section off the trapezium shape as shown into a rectangle and two triangles. The difficulty lies in knowing what the base length of each triangle is. Some students will apportion the 8 m equally as 4 m and 4 m, even though the shape is not symmetrical. Others will be aware that they don’t know these distances and it would be worth referring them back to Sandra’s method. You could even offer scissors so they can cut off the triangles and put them back together.

Some may re-allot the top part of the triangles to create two small rectangles of half the height.

An alternative strategy is to split the trapezium into two triangles across a diagonal. This relies on the students recognising what the area of the resulting triangles will be.

Total area = $504 \text{ m}^2$

Helen and Nigel work for the council. Their job is to work out the cost of surfacing playgrounds.

Below is a possible design for a new playground to be built close to the city centre.

The cheapest cost they can find for a ‘spongy’ tarmac is £20 for a square metre.

a) Make a tracing of the outline of the playground.

b) Experiment with ways of working out the cost of surfacing the whole area in ‘spongy’ tarmac. Try a few ways of doing this.

Sometimes it is cheaper to use ‘spongy’ tarmac just for the areas below the equipment and hard tarmac in between. Helen and Nigel investigate the cost of this. These areas can be seen in your workbook in Workbook exercise 1.7.

Turn to pages 12–14 in your workbook and try Workbook exercise 1.7.
Workbook exercise 1.7

Strategies will be similar to those discussed in question 19:

1. $3 \times 6 = 18 \text{ m}^2$  
   \[ \frac{1}{2}(4 \times 6) = 12 \text{ m}^2 \]  
   $18 + 12 = 30 \text{ m}^2$

2. $8 \times 2 = 16 \text{ m}^2$  
   \[ \frac{1}{2}(1 \times 8) = 4 \text{ m}^2 \]  
   $16 + 4 = 20 \text{ m}^2$

3. $6 \times 4 = 24 \text{ m}^2$  
   \[ \frac{1}{2}(1 \times 4) = 2 \text{ m}^2 \]  
   $24 - 2 = 22 \text{ m}^2$

4. Students may re-allot one of the semicircles into the gap. Encourage estimation at this stage for the bottom two semicircles, i.e. they could be approximated as two small rectangles, $4 \text{ m}$ across and $1\frac{1}{2} \text{ m}$ tall. Avoid getting bogged down with the area of a circle at this stage.

5. $8 \times 4 = 32 \text{ m}^2$  
   \[ 4 \times 1\frac{1}{2} = 6 \text{ m}^2 \]  
   $6 + 6 = 12 \text{ m}^2$

   $32 + 12 = 44 \text{ m}^2$ (or somewhere close to this)

6. $5 \times 9 = 45 \text{ m}^2$  
   \[ \frac{1}{2}(3 \times 9) = 13\frac{1}{2} \text{ m}^2 \]  
   $45 + 13\frac{1}{2} = 58\frac{1}{2} \text{ m}^2$

   $\frac{1}{2}(3\frac{1}{2} \times 2) = \frac{1}{2}(6 + 1) = 3\frac{1}{2} \text{ m}^2$

   (Students may re-allot top half of triangle and see the area as $3\frac{1}{2} \text{ by } 1 = 3\frac{1}{2}$)

   $28 + 3\frac{1}{2} = 31\frac{1}{2} \text{ m}^2$

7. The arrows may need clarification.

   Re-allotting the right-hand triangle gives a rectangle: $9 \times 8 = 72 \text{ m}^2$

8. This problem is particularly challenging as it requires students to bring together a number of the strategies used in previous sections.

   Students may split down a diagonal. Offer the image of cutting down the split and pulling the triangles apart. This may prompt some of them to see the shapes as triangles in their own right. Even then they may need to be prompted about where each triangle has come from.

   **Question 17** offered the idea that shearing (moving along) a trapezium while keeping the height fixed does not affect the area, so the trapezium in Workbook exercise 1.7 **question 8** could be seen as a right-angled trapezium.

   Area = $\frac{1}{2}(4 \times 13) + \frac{1}{2}(6 \times 13) = 78 \text{ m}^2$
Summary

Some teachers have found it useful to get the students to carefully copy out the Summary diagrams and then write their own descriptions of what each picture tells them.

In this chapter you looked at ways of finding the area of various shapes:

By ‘realloting’ parts of the shape.

By creating a surrounding rectangle.

By seeing how many squares you could fit into each shape.