Revision of GCSE vector methods

Lesson summary

Prerequisites

- Understanding of 2D trigonometry.
- Pythagoras's theorem.
- A basic understanding of velocity and acceleration.

Aim

- To revise GCSE vector theory.
- To introduce **i**, **j** notation.

Discussion

- A vector has length and direction.
- The sum of two vectors is formed by placing one on the end of the other.
- Scalar multiples of vectors are parallel.
- Column vectors can be used to describe position of points relative to an origin O.
- Length of column vectors.
- Definition of **i** and **j**.
- Definition of unit vectors and their uses.

Answers to exercises

1 **(i)** $2\mathbf{p}-4\mathbf{q}$, $4\mathbf{p}-8\mathbf{q}$
(ii) collinear

2 **(i)** (a) $\mathbf{a}+\mathbf{b}$
(b) $\mathbf{b}-\mathbf{a}$
(c) $\frac{2}{3}\mathbf{b}$
(d) $\mathbf{a}+\frac{2}{3}\mathbf{b}$
(e) $\frac{2}{3}\mathbf{a}$
(f) $\mathbf{b}+\frac{2}{3}\mathbf{a}$
(g) $\frac{1}{3}\mathbf{a}-\frac{1}{3}\mathbf{b}$

(ii) $\overrightarrow{AB} = 3\overrightarrow{NM}$ and $\overrightarrow{BM}$ is not parallel to $\overrightarrow{AN}$.

3 $(2, 6)$

4 D $(0, 16)$

5 $x = -1$, $y = -3$

6 $\frac{7}{25}\mathbf{i} + \frac{24}{25}\mathbf{j}$

7 $6\mathbf{i}+8\mathbf{j}$

8 $\lambda = 10$ or -8

9 $\lambda = 2$, $\mu = -1$

10 $\mathbf{a} = \begin{pmatrix} 14 \\ 7 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

Revision of GCSE vector methods

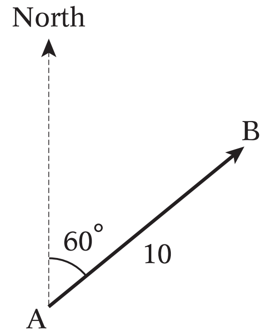
A vector can be represented by an arrow — it has **length** and **direction**. It can be used to describe force, acceleration, velocity, position and journeys, among many other things. When referring to vectors we use bold letters or use the points describing where they start and end (with an arrow above, e.g. $\overrightarrow{AC}$).

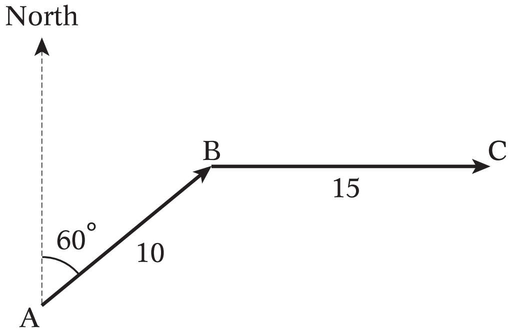
Addition

A ship is travelling at 10 km h^{-1} on a bearing of 060°. The vector shown represents the velocity of the ship.

The vector also represents the displacement of the ship in 1 hour. If the ship's starting point is A then after 1 hour it will end up at B as shown (a distance of 10 km from A). We describe the vector as $\overrightarrow{AB}$.

Suppose now that the ship changes course and travels at 15 km h^{-1} due east and finally ends up at *C*.

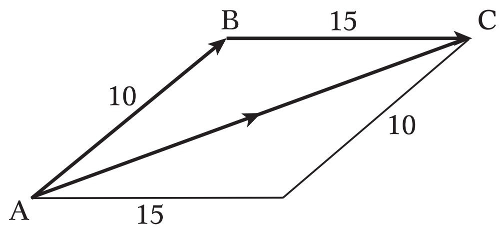


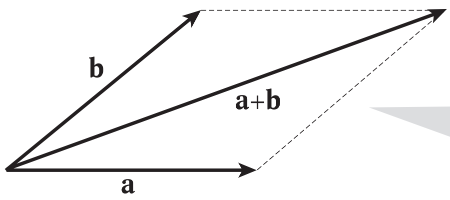


The vector $\overrightarrow{AB}$ shown above represents this part of the ship's journey.

The sum of the two journeys is a single vector $\overrightarrow{AC}$. This single journey is equivalent to the two separate journeys put together.

(Notice that if the ship first travels at 15 km h^{-1} due east for 1 hour and then at 10 km h^{-1} on a bearing of 060° for 1 hour then the ship will still end up at C.)

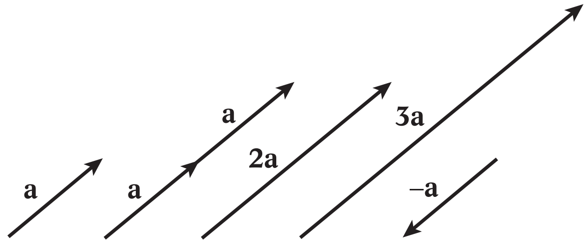




In general two vectors are added by placing one vector on the end of the other.

Scalar multiples

If we add any vector **a** to itself, we get a vector parallel to **a** and twice as long, so we can write **a** + **a** = 2**a**.

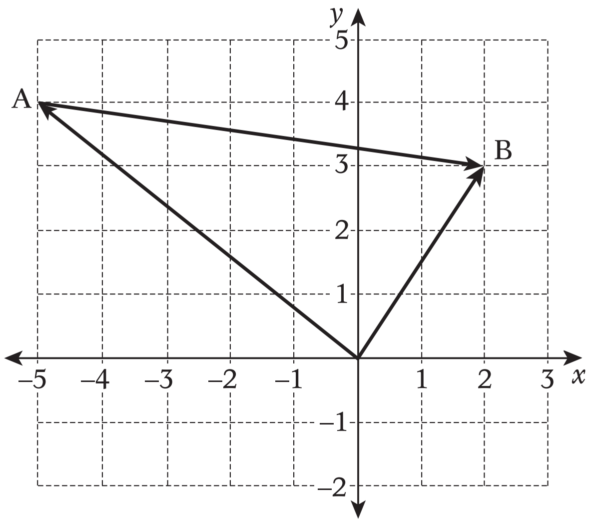


Similarly, 3**a** will be a vector parallel to **a**, but three times as long.

The vector –**a** will also be parallel to **a**, but in the reverse direction. This is because if we add **a** to (–**a**), we get 0.

Column vectors

We can describe the vector $\overrightarrow{AB}$ as a column vector when we know the coordinates of A and B. The diagram shows points $A\,(-5, 4)$ and $B\,(2, 3)$.



$$\overrightarrow{AB} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \text{ and } \overrightarrow{OA} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$$

Column vectors are simply added by summing the numbers in corresponding places.

$$\overrightarrow{OA} + \overrightarrow{AB} = \begin{pmatrix} -5 \\ 4 \end{pmatrix} + \begin{pmatrix} 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$
$$= \overrightarrow{OB}$$

Scalar multiples of column vectors are found by multiplying each element by the multiple.

$$4\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \times 2 \\ 4 \times 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

Position vectors

The vector from the origin to a particular point is called the position vector of that point. In the diagram above, we see that the position vector of A is $\overrightarrow{OA} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$.

Magnitude

The length of a vector is called its magnitude. The magnitudes of column vectors can be found using Pythagoras's theorem. The magnitude of a vector **b** (written in **bold**) is usually written as b (written in *italic*). The magnitude of $\overrightarrow{OA}$ is usually written OA or $|\overrightarrow{OA}|$.

In the diagram above, $OA = \sqrt{(-5)^2 + 4^2} = 6.40$, and $AB = \sqrt{7^2 + (-1)^2} = 7.07$.

Unit vectors

For example, consider $\mathbf{b} = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$: $b = \sqrt{0.6^2 + 0.8^2} = 1$

so **b** is a unit vector.

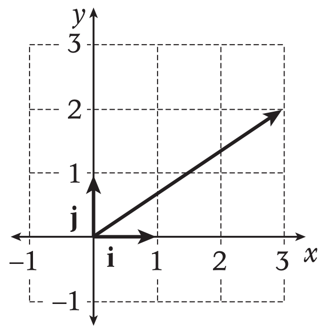
A unit vector is any vector whose magnitude is 1.

This means that 2**b** has length 2; 3**b** has length 3, and so on.

i, j notation

The vector **i** is defined as $\begin{pmatrix}1\\0\end{pmatrix}$ and **j** is defined as $\begin{pmatrix}0\\1\end{pmatrix}$.
These are special vectors on two counts:

- they are parallel to the x-axis and y-axis, respectively
- they are unit vectors.



Column vectors can be expressed in terms of **i** and **j**.

For example, the vector $\begin{pmatrix}3\\2\end{pmatrix} = 3\mathbf{i} + 2\mathbf{j}$.

The magnitude of $3\mathbf{i} + 2\mathbf{j}$ is written $|3\mathbf{i} + 2\mathbf{j}|$. And clearly from Pythagoras's theorem $|3\mathbf{i} + 2\mathbf{j}| = \sqrt{3^2 + 2^2}$.

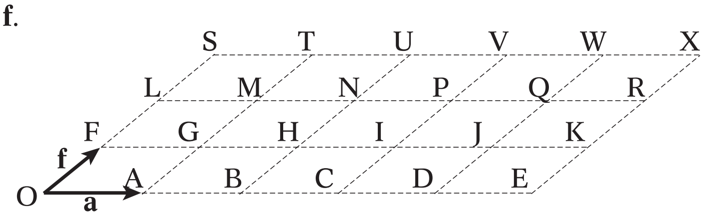
In general:

$$|a\mathbf{i} + b\mathbf{j}| = \sqrt{a^2 + b^2}$$

Revision of GCSE vector methods

Worked example 1 (to be done aurally)

The diagram below shows two sets of parallel lines. The points A, B, …, X are the points of intersection of the lines. $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OF} = \mathbf{f}$. Find the following vectors in terms of **a** and **f**.



Answer 1

$\overrightarrow{AB} = \mathbf{a}$	$\overrightarrow{WD} = -3\mathbf{f}$	$\overrightarrow{AL} = -\mathbf{a} + 2\mathbf{f}$	$\overrightarrow{VB} = -\mathbf{a} - 3\mathbf{f}$
$\overrightarrow{OB} = 2\mathbf{a}$	$\overrightarrow{OG} = \mathbf{a} + \mathbf{f}$	$\overrightarrow{CG} = -2\mathbf{a} + \mathbf{f}$	$\overrightarrow{PE} = 2\mathbf{a} - 2\mathbf{f}$
$\overrightarrow{CO} = -3\mathbf{a}$	$\overrightarrow{OM} = \mathbf{a} + 2\mathbf{f}$	$\overrightarrow{SR} = 5\mathbf{a} - \mathbf{f}$	$\overrightarrow{TF} = -\mathbf{a} - 2\mathbf{f}$
$\overrightarrow{XS} = -5\mathbf{a}$	$\overrightarrow{OT} = \mathbf{a} + 3\mathbf{f}$	$\overrightarrow{MH} = \mathbf{a} - \mathbf{f}$	$\overrightarrow{SE} = 5\mathbf{a} - 3\mathbf{f}$
$\overrightarrow{CP} = 2\mathbf{f}$	$\overrightarrow{GW} = 3\mathbf{a} + 2\mathbf{f}$	$\overrightarrow{KD} = -\mathbf{a} - \mathbf{f}$	$\overrightarrow{JL} = -4\mathbf{a} + \mathbf{f}$

Worked example 2

Show that MCJP in the diagram above is a trapezium.

Answer 2

$\overrightarrow{MC} = 2\mathbf{a} - 2\mathbf{f}$ and $\overrightarrow{PJ} = \mathbf{a} - \mathbf{f}$

$\therefore\ \overrightarrow{MC} = 2\overrightarrow{PJ}$

therefore MC is parallel to PJ.

$\overrightarrow{MP} = 2\mathbf{a}$ and $\overrightarrow{CJ} = \mathbf{a} + \mathbf{f}$; these are not multiples of each other, so they are not parallel.

$\therefore$ MCJP has one pair of parallel lines and is therefore a trapezium.

Worked example 3

Find a vector parallel to $5\mathbf{i}+12\mathbf{j}$ whose length is 39.

Answer 3

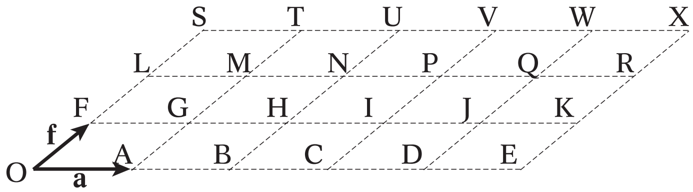
Firstly, $|5\mathbf{i} + 12\mathbf{j}| = \sqrt{25 + 144} = \sqrt{169} = 13$. The length of $(5\mathbf{i}+12\mathbf{j})$ is 13.

Hence, $3 \times (5\mathbf{i}+12\mathbf{j}) = 15\mathbf{i}+36\mathbf{j}$, will have the required length.

Revision of GCSE vector methods

Worked example 1 (to be done aurally)

The diagram below shows two sets of parallel lines. The points A, B, …, X are the points of intersection of the lines. $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OF} = \mathbf{f}$. Find the following vectors in terms of **a** and **f**.



$\overrightarrow{AB} =$	$\overrightarrow{WD} =$	$\overrightarrow{AL} =$	$\overrightarrow{VB} =$
$\overrightarrow{OB} =$	$\overrightarrow{OG} =$	$\overrightarrow{CG} =$	$\overrightarrow{PE} =$
$\overrightarrow{CO} =$	$\overrightarrow{OM} =$	$\overrightarrow{SR} =$	$\overrightarrow{TF} =$
$\overrightarrow{XS} =$	$\overrightarrow{OT} =$	$\overrightarrow{MH} =$	$\overrightarrow{SE} =$
$\overrightarrow{CP} =$	$\overrightarrow{GW} =$	$\overrightarrow{KD} =$	$\overrightarrow{JL} =$

Worked example 2

Show that MCJP in the diagram above is a trapezium.

Worked example 3

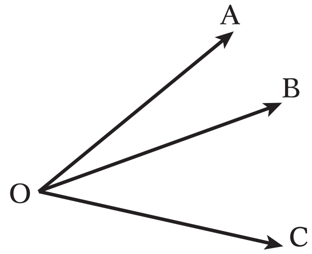
Find a vector parallel to $5\mathbf{i}+12\mathbf{j}$ whose length is 39.

Revision of GCSE vector methods

1 The diagram shows three points A, B and C with position vectors $2\mathbf{p}+\mathbf{q}$, $4\mathbf{p}-3\mathbf{q}$ and $8\mathbf{p}-11\mathbf{q}$, respectively, relative to an origin O.

(i) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

(ii) What does your answer tell you about the points A, B and C.



2 In the parallelogram OACB, shown below, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. $BM = 2MC$ and $AN = 2NC$.

(i) Find the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$.

(a) $\overrightarrow{OC}$

(b) $\overrightarrow{AB}$

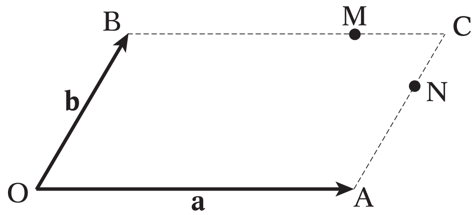
(c) $\overrightarrow{AN}$

(d) $\overrightarrow{ON}$

(e) $\overrightarrow{BM}$

(f) $\overrightarrow{OM}$

(g) $\overrightarrow{MN}$



(ii) Explain why ABMN is a trapezium.

3 Given $\overrightarrow{OA} = \begin{pmatrix}3\\2\end{pmatrix}$ and $\overrightarrow{AB} = \begin{pmatrix}-1\\4\end{pmatrix}$ find the coordinates of B.

4 Given $\overrightarrow{OA} = \begin{pmatrix}-2\\8\end{pmatrix}$, $\overrightarrow{AB} = \begin{pmatrix}3\\-3\end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix}5\\5\end{pmatrix}$, draw a diagram showing the points A, B and C. Find the coordinates of D such that ABCD is a parallelogram.

5 The sum of $\begin{pmatrix}3\\-1\end{pmatrix}$, $\begin{pmatrix}1\\2\end{pmatrix}$, $\begin{pmatrix}-3\\1\end{pmatrix}$, $\begin{pmatrix}4\\x\end{pmatrix}$ and $\begin{pmatrix}y\\2\end{pmatrix}$ is $\begin{pmatrix}2\\3\end{pmatrix}$. Find x and y.

6 Find a unit vector parallel to $7\mathbf{i} + 24\mathbf{j}$.

7 Find a vector whose magnitude is 10 and which is parallel to $3\mathbf{i} + 4\mathbf{j}$.

8 If $|(\lambda - 1)\mathbf{i} + 12\mathbf{j}| = 15$, find the possible values of λ.

9 If $\lambda\begin{pmatrix}3\\2\end{pmatrix} + \mu\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}5\\5\end{pmatrix}$, find λ and μ.

10 The vector $\mathbf{a}$ is parallel to $\begin{pmatrix}2\\1\end{pmatrix}$ and $\mathbf{b}$ is parallel to $\begin{pmatrix}-1\\1\end{pmatrix}$. Find $\mathbf{a}$ and $\mathbf{b}$ if $\mathbf{a}+\mathbf{b} = \begin{pmatrix}17\\4\end{pmatrix}$.

The **i, j, k** system

Lesson summary

Prerequisites

- GCSE vectors.
- 2D trigonometry.
- Pythagoras's theorem.

Aim

- Introduction of **i**, **j**, **k**.
- Uses of vectors to find distances and angles in 3D.

Discussion

- Coordinates in 3D and definition of **k**.
- Magnitude of a vector.
- Distance between points.
- Use of sine and cosine rules.

Answers to exercises

1 **(i)** 3.74 **(ii)** 2.24 **(iii)** 9 **(iv)** 5.39
(v) 24.8 **(vi)** 3.74

2 **(i)** $\frac{1}{9}(8\mathbf{i}-\mathbf{j}+4\mathbf{k})$ **(ii)** $\frac{1}{25}(7\mathbf{j}-24\mathbf{k})$ **(iii)** $\frac{1}{3}(2\mathbf{i}+\mathbf{j}-2\mathbf{k})$

3 **(i)** $\frac{10}{9}(8\mathbf{i}-\mathbf{j}+4\mathbf{k})$ **(ii)** $\frac{2}{5}(7\mathbf{j}-24\mathbf{k})$ **(iii)** $\frac{10}{3}(2\mathbf{i}+\mathbf{j}-2\mathbf{k})$

4 **(i)** $4\mathbf{j}$ **(ii)** $6\mathbf{i}+2\mathbf{k}$ **(iii)** $-3\mathbf{i}+6\mathbf{j}-\mathbf{k}$ **(iv)** 4

6 10 or –2

7 5.74

9 $(1, 1, 4)$

10 90°

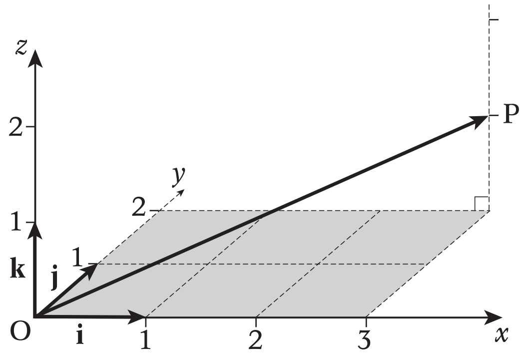
11 **(ii)** $(4, 2, -1)$
(iii) $8\mathbf{i}+4\mathbf{j}-2\mathbf{k}$

12 4.33

The i, j, k system

In two dimensions we have seen that the position vectors of points can be given in terms of **i** and **j**, where **i** and **j** are unit vectors parallel to the x-axis and y-axis, respectively.

Points in three dimensions have three coordinates (x, y, z). We need three vectors to specify position vectors. We define **k** as the unit vector parallel to the z-axis as shown.



The position vector of P in the above diagram is $3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ or $\begin{pmatrix}3\\2\\1\end{pmatrix}$.

The magnitude of the vector $\overrightarrow{OP}$ can be found using Pythagoras's theorem:

$$OP = \sqrt{3^2 + 2^2 + 1^2} = 3.74$$

In general:

$$|a\mathbf{i} + b\mathbf{j} + c\mathbf{k}| = \sqrt{a^2 + b^2 + c^2}$$

This formula gives us a useful way of finding the distance between points in three dimensions.

The i, j, k system

Worked example 1

Show that the triangle A(2, 4, –1), B(1, 0, 2) and C(2, 5, 2) is isosceles.

Answer 1

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \text{ and } \overrightarrow{OC} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} \Rightarrow \overrightarrow{AB} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix}$$

Similarly, $\overrightarrow{BC} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$ and $\overrightarrow{CA} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$

$AB = \sqrt{(-1)^2 + (-4)^2 + 3^2} = \sqrt{26}$, $BC = \sqrt{1^2 + 5^2 + 0^2} = \sqrt{26}$

and $CA = \sqrt{0^2 + (-1)^2 + (-3)^2} = \sqrt{10}$

$AB = BC$ so the triangle is isosceles.

Worked example 2

Find a unit vector parallel to $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

Answer 2

$|\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}| = \sqrt{1 + 4 + 4} = 3 \Rightarrow \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ is a unit vector.

Worked example 3

The points A(2, 4, –1), B(1, 0, 2), C(2, 5, 2) and D form a parallelogram. Find the coordinates of D.

Answer 3

If ABCD is a parallelogram then $\overleftrightarrow{AB} = \overrightarrow{CD}$.

From the above coordinates, $\overrightarrow{BA} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ so $\overrightarrow{CD} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$.

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ -1 \end{pmatrix}.$$

Hence D is the point (3, 9, –1).

Worked example 4

Find ∠ABC in the triangle where the coordinates of A, B and C are given by A(2, 4, –1), B(1, 0, 2) and C(2, 5, 2) (the same coordinates as Worked example 1).

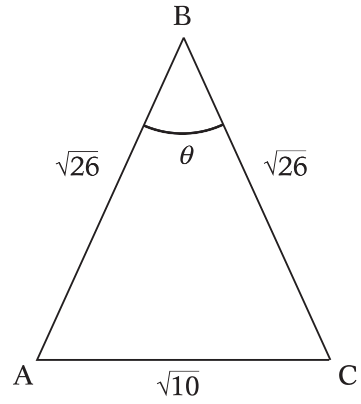
Answer 4

From the results in Worked example 1, $AB = \sqrt{26} = BC$ and $CA = \sqrt{10}$.

From the cosine rule in triangle ABC:

$$\cos\theta = \frac{26 + 26 - 10}{2 \times 26} = 0.808$$

so $\theta = 36.1°$ (to 3 sig. figs.)



The i, j, k system

Worked example 1

Show that the triangle A(2, 4, –1), B(1, 0, 2) and C(2, 5, 2) is isosceles.

Worked example 2

Find a unit vector parallel to $\mathbf{i}-2\mathbf{j}+2\mathbf{k}$.

Worked example 3

The points A(2, 4, –1), B(1, 0, 2), C(2, 5, 2) and D form a parallelogram. Find the coordinates of D.

Worked example 4

Find ∠ABC in the triangle where the coordinates of A, B and C are given by A(2, 4, –1), B(1, 0, 2) and C(2, 5, 2) (the same coordinates as Worked example 1).

The i, j, k system

1 Find the magnitudes of the following vectors, correct to 3 sig. fig.
(i) $3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ **(ii)** $2\mathbf{i}-\mathbf{k}$ **(iii)** $8\mathbf{i}-\mathbf{j}+4\mathbf{k}$ **(iv)** $5\mathbf{j}+2\mathbf{k}$
(v) $6\mathbf{i}-24\mathbf{j}+\mathbf{k}$ **(vi)** $-3\mathbf{i}+2\mathbf{j}-\mathbf{k}$

2 Find unit vectors parallel to the following:
(i) $8\mathbf{i}-\mathbf{j}+4\mathbf{k}$ **(ii)** $7\mathbf{j}-24\mathbf{k}$ **(iii)** $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$

3 Find vectors of magnitude 10 parallel to the following:
(i) $8\mathbf{i}-\mathbf{j}+4\mathbf{k}$ **(ii)** $7\mathbf{j}-24\mathbf{k}$ **(iii)** $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$

4 If $\mathbf{a} = 3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $\mathbf{b} = -3\mathbf{i}+2\mathbf{j}-\mathbf{k}$ find:
(i) $\mathbf{a}+\mathbf{b}$ **(ii)** $\mathbf{a}-\mathbf{b}$ **(iii)** $\mathbf{a}+2\mathbf{b}$ **(iv)** $|\mathbf{a}+\mathbf{b}|$

5 Show that the points A(3, 0, 1), B(7, –4, 2) and C(–5, 8, –1) are collinear.

6 The magnitude of the vector $[\mathbf{i}+(\lambda-2)\mathbf{j}-(\lambda-6)\mathbf{k}]$ is 9. Find two possible values of λ.

7 Find the distance between the points A(3, 0, 1) and B(7, –4, 2).

8 Show that the triangle ABC is right-angled where A(3, –2, 4), B(4, 0, 3) and C(5, –2, 6).

9 The points A(2, 0, 2), B(1, 2, –1), C(0, 3, 1) and D form a parallelogram. Find the coordinates of D.

10 Find $\angle ABC$ in the triangle where the coordinates of A, B and C are given by A(4, 2, 1), B(1, 0, 2) and C(2, 1, 7).

11 The position vectors of points P and Q, relative to an origin O, are **p** and **q**, respectively.

(i) Show that the position vector of the mid-point M of PQ is given by $\overrightarrow{OM} = \dfrac{\mathbf{p}+\mathbf{q}}{2}$.

(ii) Find the coordinates of the mid-point of PQ where $\overrightarrow{OP} = 2\mathbf{i}+5\mathbf{j}-3\mathbf{k}$ and $\overrightarrow{OQ} = 6\mathbf{i}-\mathbf{j}+\mathbf{k}$.

(iii) Find the position vector of R, such that OPRQ is a parallelogram.

(iv) Show that the quadrilateral formed by joining the mid-points of the parallelogram OPRQ is also a parallelogram.

12 Find the area of the triangle A(2, 0, 2), B(1, 2, –1), C(0, 3, 1).

suvat formulae in 2D and 3D

Lesson summary

Prerequisites

- *suvat* equations for motion of a particle in 1D.
- GCSE vector arithmetic.
- **i**, **j**, **k** notation.

Aims

- Apply the constant acceleration formulae in 2D and 3D.

Discussion

- Give an example describing the position vector of a particle travelling with constant velocity.
- Give the *suvat* formulae in vector form.

Answers to exercises

1 **(i)** $\mathbf{v} = 15\mathbf{i} + 22\mathbf{j}$
(ii) $26.8\,\text{m s}^{-1}$

2 **(i)** $\mathbf{v} = (6 - 2t)\mathbf{i} + (3 + t)\mathbf{j}$
(ii) $t = 3\,\text{s}$
(iii) $t = 1\,\text{s}$

3 **(i)** $-0.3\mathbf{i} + 0.2\mathbf{j}$
(ii) $\mathbf{r} = (2t - 0.15t^2 + 20)\mathbf{i} + (0.1t^2 - t)\mathbf{j}$
(iii) (b) $5\,\text{m s}^{-1}$

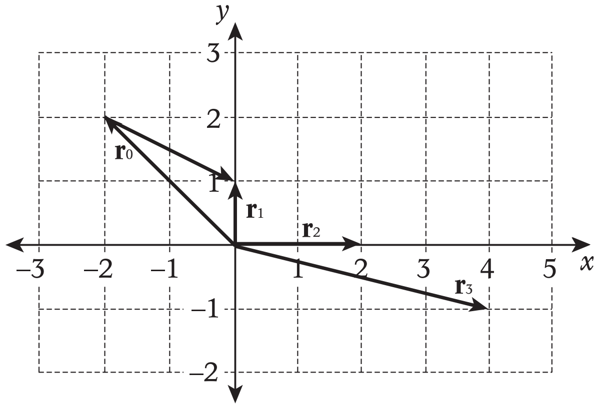
4 **(i)** $3\mathbf{i}$
(ii) $\mathbf{r} = (3t - 0.1t^2)\mathbf{i} - 0.15t^2\mathbf{j}$
(iii) 30 s
(iv) 15.4 m

5 **(i)** 031°
(ii) $\mathbf{a} = 9t\mathbf{j}$, $\mathbf{b} = (-10 + 3t)\mathbf{i} + 5t\mathbf{j}$
(iii) 1520 hours
(v) 1424 hours

suvat formulae in 2D and 3D

Consider a particle which starts from a point (–2, 2) m and moves with a velocity of $\begin{pmatrix}2\\-1\end{pmatrix}$ m s^{-1}.

The diagram shows the starting point and the position of the particle after 1, 2 and 3 seconds.



The velocity vector is the change in displacement in one unit of time. The speed of the particle is the distance travelled in one unit of time. The speed is the length of the velocity vector. In this example the speed of the particle is $\sqrt{2^2+(-1)^2} = \sqrt{5}$ m s^{-1}.

The position of the particle can be written in terms of the starting point and the velocity vector.

- After 1 second, the position vector of the particle is $\mathbf{r}_1 = \begin{pmatrix}-2\\2\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$.
- After 2 seconds, the position vector of the particle is $\mathbf{r}_2 = \begin{pmatrix}-2\\2\end{pmatrix} + 2\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$.
- After 3 seconds, the position vector of the particle is $\mathbf{r}_3 = \begin{pmatrix}-2\\2\end{pmatrix} + 3\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}4\\-1\end{pmatrix}$.

So, at any point in time we can write the position vector of the particle as:

$$\mathbf{r}_t = \begin{pmatrix}-2\\2\end{pmatrix} + t\begin{pmatrix}2\\-1\end{pmatrix}.$$

In general, the position vector, **r**, of a particle which starts at the point with position vector $\mathbf{r}_0$ and moves with a constant velocity **v** will be:

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$$

The above formula also applies to motion in two dimensions and three dimensions.

In the same way as velocity is the rate of change of displacement, acceleration is the rate of change of velocity.

Hence, the velocity, $\mathbf{v}$, of a particle which starts with velocity $\mathbf{u}$ and moves with constant acceleration $\mathbf{a}$ will be:

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

In fact, the methods used to derive the *suvat* equations in one dimension can be used to derive the following formulae which can be used in two and three dimensions.

$$\mathbf{r} = \mathbf{u}t + \tfrac{1}{2}\mathbf{a}t^2,\ \mathbf{r} = \left(\frac{\mathbf{u}+\mathbf{v}}{2}\right)t \text{ and } \mathbf{r} = \mathbf{v}t - \tfrac{1}{2}\mathbf{a}t^2$$

These formulae relate to a particle that starts at the origin, undergoes a displacement $\mathbf{r}$ in time t, has an initial velocity $\mathbf{u}$, a final velocity $\mathbf{v}$ and a constant acceleration $\mathbf{a}$.

If the particle starts at the point with position vector $\mathbf{r}_0$ instead, then these formulae become:

$$\mathbf{r} = \mathbf{r_0} + \mathbf{u}t + \tfrac{1}{2}\mathbf{a}t^2$$

$$\mathbf{r} = \mathbf{r_0} + \left(\frac{\mathbf{u}+\mathbf{v}}{2}\right)t$$

$$\mathbf{r} = \mathbf{r_0} + \mathbf{v}t - \tfrac{1}{2}\mathbf{a}t^2$$

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

suvat formulae in 2D and 3D

Worked example 1

A particle, P, moves with acceleration $(3\mathbf{i}+\mathbf{j})\,\text{m s}^{-2}$. The initial velocity of the particle is $4\mathbf{j}\,\text{m s}^{-1}$. Find the velocity of the particle after t seconds and the speed of the particle after 1 second.

Answer 1

Using '$\mathbf{v} = \mathbf{u} + \mathbf{a}t$', $\mathbf{v} = 4\mathbf{j} + (3\mathbf{i}+\mathbf{j})t = 3t\mathbf{i} + (4+t)\mathbf{j}$

When $t = 1$, $\mathbf{v} = 3\mathbf{i}+5\mathbf{j}$ and $|3\mathbf{i} + 5\mathbf{j}| = \sqrt{3^2 + 5^2} = \sqrt{34} = 5.83\,\text{m s}^{-1}$

Worked example 2

The position vectors of two ships P and Q at midday are given by $\begin{pmatrix}2\\1\end{pmatrix}$ km and $\begin{pmatrix}4\\10\end{pmatrix}$ km, relative to an origin O. The ships P and Q move with constant velocities $\begin{pmatrix}-1\\1\end{pmatrix}$ km h^{-1} and $\begin{pmatrix}-3\\-1\end{pmatrix}$ km h^{-1}, respectively. Find the shortest distance between the two ships in the ensuing motion.

Answer 2

The position vectors of P and Q at time t hours after midday are:

$$\mathbf{r}_P = \begin{pmatrix}2\\1\end{pmatrix} + t\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2-t\\1+t\end{pmatrix} \text{ and } \mathbf{r}_Q = \begin{pmatrix}4\\10\end{pmatrix} + t\begin{pmatrix}-3\\-1\end{pmatrix} = \begin{pmatrix}4-3t\\10-t\end{pmatrix}$$

$$\overrightarrow{PQ} = \mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}4-3t\\10-t\end{pmatrix} - \begin{pmatrix}2-t\\1+t\end{pmatrix} = \begin{pmatrix}2-2t\\9-2t\end{pmatrix}$$

The distance between the particles is given by the magnitude of the vector.

$$PQ^2 = (2-2t)^2 + (9-2t)^2 = 8t^2 - 44t + 85$$

The shortest distance between the two ships is given by the minimum value of this expression, which can be found using calculus or completing the square.

Completing the square gives $PQ^2 = 8t^2 - 44t + 85 = 8\left(t - \frac{11}{4}\right)^2 + 24.5$.

The minimum value of PQ is $\sqrt{24.5} = 4.95$ km when $t = 2.75$ h

suvat formulae in 2D and 3D

Worked example 1

A particle, P, moves with acceleration $(3\mathbf{i}+\mathbf{j})\,\mathrm{m\,s^{-2}}$. The initial velocity of the particle is $4\mathbf{j}\,\mathrm{m\,s^{-1}}$. Find the velocity of the particle after t seconds and the speed of the particle after 1 second.

Worked example 2

The position vectors of two ships P and Q at midday are given by $\begin{pmatrix}2\\1\end{pmatrix}$km and $\begin{pmatrix}4\\10\end{pmatrix}$km, relative to an origin O. The ships P and Q move with constant velocities $\begin{pmatrix}-1\\1\end{pmatrix}\,\mathrm{km\,h^{-1}}$ and $\begin{pmatrix}-3\\-1\end{pmatrix}\,\mathrm{km\,h^{-1}}$, respectively. Find the shortest distance between the two ships in the ensuing motion.

suvat formulae in 2D and 3D

1 A particle P moves with constant acceleration $(3\mathbf{i}+4\mathbf{j})\,\text{m s}^{-2}$. The initial velocity of P is $2\mathbf{j}\,\text{m s}^{-1}$.
(i) Find the velocity of P 5 seconds later.
(ii) Find the speed of P 5 seconds later.

2 A particle P moves with constant acceleration $(-2\mathbf{i}+\mathbf{j})\,\text{m s}^{-2}$. Initially the velocity of the particle is $(6\mathbf{i}+3\mathbf{j})\,\text{m s}^{-1}$.
(i) Find an expression for the velocity of P at time t seconds.
(ii) Find the time when P is travelling in the $\mathbf{i}$ direction.
(iii) Find the time when P is travelling parallel to the line $y = x$.

[In Questions 3 to 5, the horizontal unit vectors **i** and **j** are due east and due north, respectively.]

3 The unit vectors **i** and **j** are directed east and north respectively. A yacht moves with constant acceleration. At time t seconds, the position vector of the particle is **r** metres. When $t = 0$ the velocity of the yacht is $(2\mathbf{i}-\mathbf{j})\,\text{m s}^{-1}$ and when $t = 10$ the velocity of the yacht is $(-\mathbf{i}+\mathbf{j})\,\text{m s}^{-1}$.
(i) Find the acceleration of the yacht.
(ii) When $t = 0$ the yacht is 20 m due east of the origin. Find an expression for **r** in terms of t.
(iii) (a) Show that when $t = 20$ the yacht is due north of the origin.
(b) Find the speed of the yacht when $t = 20$. AQA (January 2005)

4 At time $t = 0$, a boat is at the origin travelling due east with speed $3\,\text{m s}^{-1}$. The boat experiences a constant acceleration of $(-0.2\mathbf{i}-0.3\mathbf{j})\,\text{m s}^{-2}$.
(i) Write down the initial velocity of the boat.
(ii) Find an expression for the position of the boat at time t seconds.
(iii) Find the time when the boat is due south of the origin.
(iv) Find the distance of the boat from the origin when it is travelling south-east.
AQA

5 Two boats A and B are moving with constant velocities. Boat A moves with velocity $9\mathbf{j}\,\text{km h}^{-1}$. Boat B moves with velocity $(3\mathbf{i}+5\mathbf{j})\,\text{km h}^{-1}$.
(i) Find the bearing on which B is moving.

At noon, A is at point O, and B is 10 km due west of O. At time t hours after noon, the position vectors of A and B relative to O are **a** km and **b** km, respectively.
(ii) Find expressions for **a** and **b** in terms of t, giving your answer in the form $p\mathbf{i}+q\mathbf{j}$.
(iii) Find the time when B is due south of A.

At time t hours after noon, the distance between A and B is d km. By finding an expression for $\overrightarrow{AB}$:
(iv) show that $d^2 = 25t^2 - 60t + 100$

At noon, the boats are 10 km apart.
(v) Find the time after noon at which the boats are again 10 km apart.
Edexcel (January 2004)

Adding velocities

Lesson summary

Prerequisites

- Sine and cosine rules.
- Adding velocities.

Aim

- Solve problems where two velocities are added to find the resultant velocity.

Discussion

- Describe common situations where the velocity of a particle is the resultant of two velocity vectors:
  – water currents and boats;
  – winds and aircraft.
- Relative velocity formula.

Answers to exercises

1 34.5 km h^{-1}

2 2.2 km h^{-1}

3 272 km h^{-1}, 094°

4 **(i)** 202 m s^{-1}
(ii) 081.5°

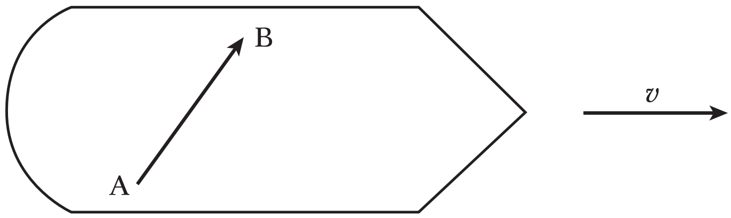
5 096°, 313 km h^{-1}, 2 hours and 24 min

6 341° or 109°

7 **(ii)** 6.74 to 6.75 m s^{-1}

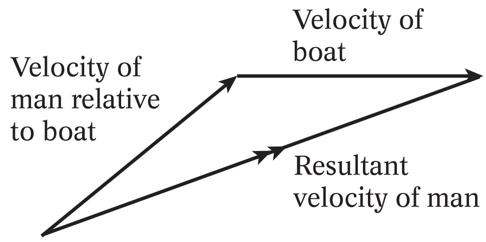
Adding velocities

Suppose a man runs across the deck of a boat from point A to point B, as shown by the vector below. His velocity relative to the boat will be a vector parallel to $\overrightarrow{AB}$.



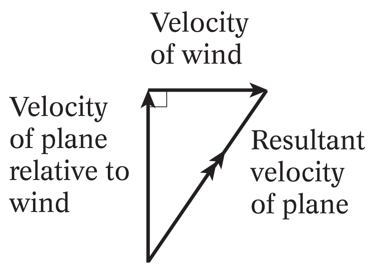
Also, suppose that the boat is moving forwards at $v\,\text{m}\,\text{s}^{-1}$.

The resultant velocity of the man is the sum of the two velocities



Similarly, suppose an aircraft pilot steers his plane to fly due north but the wind blows the plane to the east.

The resultant velocity of the plane is the sum of two velocities.



In general we can write:

velocity of A = velocity of ‘A relative to B’ + velocity of B

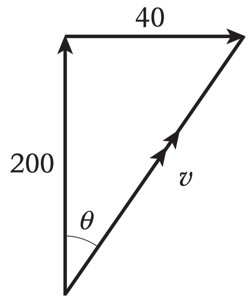
When adding two velocities which have a resultant, we need to draw a triangle of velocities.

Adding velocities

Worked example 1

An aeroplane can maintain a steady speed of 200 km h^{-1} in still air. On one particular flight it steers due north, but it is blown off course by a wind coming from the west at a speed of 40 km h^{-1}. Find the speed of the aircraft relative to the ground and the bearing on which it flies.

Answer 1



Using Pythagoras's theorem, $v^2 = 200^2 + 40^2 \Rightarrow v = 204\,\text{km h}^{-1}$

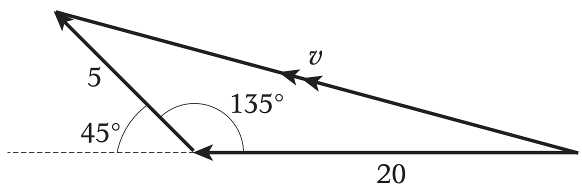
The bearing of the flight path is the angle θ.

$$\tan\theta = \frac{40}{200} \;\therefore\; \theta = 011° \text{ to the nearest degree}$$

Worked example 2

A boat that is heading west at 20 km h^{-1} is carried off course by a current of 5 km h^{-1} heading north-west. Find the resultant speed of the boat.

Answer 2



Using the cosine rule $v^2 = 20^2 + 5^2 - 2 \times 20 \times 5 \times \cos 135°$

$\Rightarrow v = 23.8\,\text{km h}^{-1}$

Adding velocities

Worked example 1

An aeroplane can maintain a steady speed of $200\,\text{km}\,\text{h}^{-1}$ in still air. On one particular flight it steers due north, but it is blown off course by a wind coming from the west at a speed of $40\,\text{km}\,\text{h}^{-1}$. Find the speed of the aircraft relative to the ground and the bearing on which it flies.

Worked example 2

A boat that is heading west at $20\,\text{km}\,\text{h}^{-1}$ is carried off course by a current of $5\,\text{km}\,\text{h}^{-1}$ heading north-west. Find the resultant speed of the boat.

Adding velocities

1 A boat that is heading north at $30\,\text{km}\,\text{h}^{-1}$ is carried off course by a current of $6\,\text{km}\,\text{h}^{-1}$ heading north-east. Find the resultant speed of the boat.

2 A swimmer crosses a river to an exact opposite point on the far bank. He swims at $2.5\,\text{km}\,\text{h}^{-1}$ relative to the water. The current flows at $1.2\,\text{km}\,\text{h}^{-1}$. Calculate his speed relative to the bank.

3 An aeroplane can maintain a steady speed of $250\,\text{km}\,\text{h}^{-1}$ relative to the air. On one particular flight it steers due east, but it is blown off course by a wind coming from the north-west at a speed of $30\,\text{km}\,\text{h}^{-1}$. Find the speed of the aircraft relative to the ground, and the bearing on which it flies.

4 An aeroplane is heading due east at $200\,\text{m}\,\text{s}^{-1}$ relative to the air. A wind is blowing due north at $30\,\text{m}\,\text{s}^{-1}$.

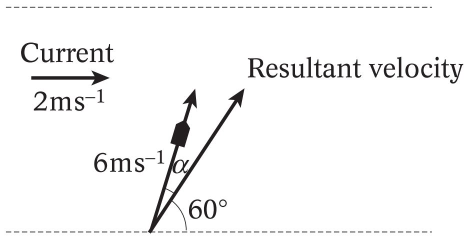
(i) Find the magnitude of the resultant velocity of the aeroplane.

(ii) Find the bearing on which the aeroplane actually flies. AQA (June 2005)

5 Two airports, A and B, are 750 km apart, with A being due east of B. A plane flies directly from A to B, at $350\,\text{km}\,\text{h}^{-1}$ relative to the air. The wind blows from the south-east at $50\,\text{km}\,\text{h}^{-1}$. Find the bearing that the pilot follows, the ground speed of the aircraft and the length of the flight.

6 A plane maintains a speed of $600\,\text{km}\,\text{h}^{-1}$ relative to the air. On a particular flight it steers north-east, but is blown off course by a wind of speed $50\,\text{km}\,\text{h}^{-1}$. The resulting ground speed of the aircraft is $580\,\text{km}\,\text{h}^{-1}$. Find the direction from which the wind is blowing.

7 A motor boat can travel at a speed of $6\,\text{m}\,\text{s}^{-1}$ relative to the water. It is used to cross a river in which the current flows at $2\,\text{m}\,\text{s}^{-1}$. The resultant velocity of the boat makes an angle of 60° to the bank, as shown in the diagram.



The angle between the direction in which the boat is travelling relative to the water and the resultant velocity is α.

(i) Show that $\alpha = 16.8°$, correct to three significant figures.

(ii) Find the magnitude of the resultant velocity. AQA (January 2005)