

# Solving equations by trial and improvement

## This chapter is about

- Finding consecutive integers between which the solution of an equation lies
- Using the method of trial and improvement to solve an equation to a given degree of accuracy

## You should already know

- How to substitute numbers into simple formulae

## Solving equations by trial and improvement

Sometimes you will be asked to solve an equation that you do not know how to solve algebraically. It might contain a number of terms, some in higher powers of  $x$ , such as  $x^3$ , or with algebraic fractions, such as  $\frac{1}{x}$ . You solve equations like this by **trial and improvement**.

This means you substitute different values into the equation until you find a solution. It is important that you work systematically and do not just choose the numbers you try at random.

Often you are told that the solution of an equation lies between two given numbers. If you are not told this, your first step must be to find these two values. Then choose the number halfway between these values. Keep making improvements until the required accuracy is achieved.

### Example 3.1

A solution of the equation  $x^3 - 4x + 1 = 0$  lies between 1 and 2.

Use trial and improvement to find the solution correct to 1 decimal place.

### Solution

You have been given two values between which the solution lies, so for the first trial use the value halfway between these, that is 1.5. A table can be used to set out your trials.

$x$ -value	Calculation	Comment
1.5	$1.5^3 - 4 \times 1.5 + 1 = -1.625$	Too small, so try a value between 1.5 and 2.0
1.8	$1.8^3 - 4 \times 1.8 + 1 = -0.368$	Too small, so try a value between 1.8 and 2.0
1.9	$1.9^3 - 4 \times 1.9 + 1 = 0.259$	Too big, so the solution lies between 1.8 and 1.9
1.85	$1.85^3 - 4 \times 1.85 + 1 = -0.0684$	Too small, so the solution lies between 1.85 and 1.9

Once you know that the solution lies between 1.8 and 1.9, you know that the solution correct to one decimal place is either 1.8 or 1.9. It only remains to decide which. To do this you try 1.85.

The solution is greater than 1.85 so is nearer to 1.9 than to 1.8.

So  $x = 1.9$ , to 1 decimal place.

**TIP**

Make sure you give the  $x$ -value (not the value on the right-hand side of the equation) to the required accuracy.

**Example 3.2**

Show that  $x^3 - 3x = 6$  has a solution between 2 and 3.

Find the solution correct to 1 decimal place.

**Solution**

$$2^3 - 3 \times 2 = 2$$

$$3^3 - 3 \times 3 = 18$$

Because 6 is between 2 and 18, there is a solution for  $x$  between 2 and 3. For the first trial use the value halfway between these, that is 2.5.

$x$ -value	Calculation	Comment
2.5	$2.5^3 - 3 \times 2.5 = 8.125$	Too big, so try a value between 2 and 2.5
2.3	$2.3^3 - 3 \times 2.3 = 5.267$	Too small, so try a value between 2.3 and 2.5
2.4	$2.4^3 - 3 \times 2.4 = 6.624$	Too big, so the solution lies between 2.3 and 2.4
2.35	$2.35^3 - 3 \times 2.35 = 5.928$	Too small, so the solution lies between 2.35 and 2.4

The solution is nearer to 2.4 than to 2.3.

So  $x = 2.4$ , to 1 decimal place.

**TIP**

Always give the result of the calculation for the trial you have done as well as saying 'too big' or 'too small'.

### Example 3.3

Show that  $\frac{6}{x} = x^2 + 3$  has a solution between 1 and 2.

Find this solution correct to 2 decimal places.

### Solution

Since  $x$  appears on both sides of this equation, it is often easier to work with the left-hand side (LHS) and right-hand side (RHS) of the equation separately and then compare them. The solution is when they are equal. When this is the case, the difference between them will be zero.

$x$ -value	LHS = $\frac{6}{x}$	RHS = $x^2 + 3$	LHS - RHS	Comment
1	$6 \div 1 = 6$	$1^2 + 3 = 4$	2	
2	$6 \div 2 = 3$	$2^2 + 3 = 7$	-4	The sign of LHS - RHS has changed, so there is a solution between 1 and 2. Now try halfway between them.
1.5	$6 \div 1.5 = 4$	$1.5^2 + 3 = 5.25$	-1.25	So the solution is between 1 and 1.5. Try 1.3.
1.3	$6 \div 1.3 = 4.615\dots$	$1.3^2 + 3 = 4.69$	-0.0746...	So the solution is between 1 and 1.3 but quite near 1.3. Try 1.2.
1.2	$6 \div 1.2 = 5$	$1.2^2 + 3 = 4.44$	0.56	So the solution is between 1.2 and 1.3 and near 1.3. Try 1.28.
1.28	$6 \div 1.28 = 4.6875$	$1.28^2 + 3 = 4.6384$	0.0491	So the solution is between 1.28 and 1.3. Try 1.29.
1.29	$6 \div 1.29 = 4.651\dots$	$1.29^2 + 3 = 4.6641$	-0.012...	So the solution is between 1.28 and 1.29. Try 1.285.
1.285	$6 \div 1.285 = 4.669\dots$	$1.285^2 + 3 = 4.651\ 225$	0.018	So the solution is between 1.285 and 1.29. So the solution is 1.29 to 2 decimal places.

The solution is  $x = 1.29$ , to 2 decimal places.

**Exercise 3.1**

- 1 (a)  $1^3 = 1$  and  $2^3 = 8$ .  
Explain how this shows that there is a solution to the equation  $x^3 = 5$  between 1 and 2.
- (b) Find the solution correct to 1 decimal place.
- 2 (a)  $2^3 - 8 \times 2 = -8$  and  $3^3 - 8 \times 3 = 3$ .  
Explain how this shows that there is a solution to the equation  $x^3 - 8x = 0$  between 2 and 3.
- (b) Find the solution correct to 1 decimal place.
- 3 (a)  $1^3 - 2 \times 1 = -1$  and  $2^3 - 2 \times 2 = 4$ .  
Explain how this shows that there is a solution to the equation  $x^3 - 2x = 0$  between 1 and 2.
- (b) Find the solution correct to 1 decimal place.
- 4 Show that a solution of  $x^3 - 5x = 8$  lies between 2 and 3.  
Find it correct to 1 decimal place.
- 5 Show that a solution of  $x^3 - x = 90$  lies between 4 and 5.  
Find it correct to 1 decimal place.
- 6 Show that a solution of  $x^3 - x^2 = 30$  lies between 3 and 4.  
Find it correct to 1 decimal place.
- 7 Show that a solution of  $x^3 - 7x = 25$  lies between 3 and 4.  
Find it correct to 1 decimal place.
- 8 Show that a solution of  $x^3 + 2x = 2$  lies between 0 and 1.  
Find it correct to 1 decimal place.
- 9 Show that a solution of  $x^3 - x^2 = 1$  lies between 1 and 2.  
Find it correct to 1 decimal place.
- 10 A solution of  $x^3 + 50 = 0$  lies between  $-4$  and  $-3$ .  
Find it correct to 1 decimal place.
- 11 A solution of  $x^3 + 4x + 25 = 0$  lies between  $-3$  and  $-2$ .  
Find it correct to 1 decimal place.
- 12 A solution of  $x^3 + 12 = 0$  lies between  $-3$  and  $-2$ .  
Find it correct to 1 decimal place.
- 13 A solution of  $x^3 - 2x + 6 = 0$  lies between  $-3$  and  $-2$ .  
Find it correct to 1 decimal place.

*Exercise continues ...*

- 14 (a) Find two consecutive integers between which the solution of  $x^3 - 2x = 4$  lies.  
 (b) Find the solution correct to 2 decimal places.
- 15 The equation  $x = \frac{2}{x^2} + 1$  has a solution between  $x = 1$  and  $x = 2$ .  
 Find this solution correct to 1 decimal place.
- 16 (a) Show that  $x = -2$  is a solution of the equation  $x^2 - 5 = \frac{2}{x}$ .  
 (b) Use trial and improvement to find, correct to 1 decimal place, the solution of this equation that is between 2 and 3.

## What you have learned

- To find the solution to an equation by trial and improvement you first need to find two numbers between which the solution lies. You then try the number halfway between these two numbers and continue the process until you find the answer to the required degree of accuracy

## Mixed exercise 3

- 1 A solution of  $x^3 + 3x^2 + x = 0$  lies between  $-3$  and  $-2$ .  
 Find it correct to 2 decimal places.
- 2 (a) Find two consecutive integers between which a solution of  $x^3 + 3x - 20 = 0$  lies.  
 (b) Find the solution correct to 2 decimal places.
- 3 A solution of  $x^3 - 5x^2 + 2x = 0$  lies between 0 and 1.  
 Find it correct to 2 decimal places.
- 4 (a) Show that a solution to the equation  $x^3 + 4x = 12$  lies between  $x = 1$  and  $x = 2$ .  
 (b) Find the solution correct to 1 decimal place.
- 5 (a) Show that a solution to the equation  $x^3 - x^2 = 28$  lies between  $x = 3$  and  $x = 4$ .  
 (b) Find the solution correct to 1 decimal place.
- 6 A number, added to the cube of this number, gives 100.  
 (a) Write this as an equation. (b) Find the number correct to 1 decimal place.
- 7 The equation  $x^2 - 8 = \frac{3}{x}$  has a solution between  $x = -2$  and  $x = -3$ .  
 Find this solution, correct to 2 decimal places.